Topology Comprehensive Exam
Fall 2020

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider $\mathbb{R}^2$ with the usual (x, y)-coordinates. Let φ be the 1-form on $\mathbb{R}^2$ given by

$$\varphi = \frac{y\,dx - x\,dy}{x^2 + 4y^2}$$

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be the smooth curve defined by

$$\gamma(t) = (2\cos(2\pi t), \sin(2\pi t)).$$

(a) Compute $d\varphi$.
(b) Compute $\int_\gamma \varphi$.
(c) Is φ closed?
(d) Is φ exact?

2. Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ denote the set of nonzero real numbers. Fix some $p \in \mathbb{R}^3$.
(a) Show that the function

$$F : \mathbb{R}^* \times \mathbb{R}^3 \to \mathbb{R}^3$$

given by

$$F(t, v) = p + tv$$

is a submersion.
(b) Use part (a) to show that almost every line through p intersects the unit sphere S^2 transversely if $p \notin S^2$.

Hint: You may use the following statement of the transversality theorem: if $F : X \times S \to Y$ is transverse to $Z \subseteq Y$ then for almost every $s \in S$ the map $f_s : X \times \{s\} \to Y$ is transverse to Z.

3. Let $M_3(\mathbb{R})$ denote the space of real-valued 3×3 matrices. Let M be the 3×3 diagonal matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Let G denote the set of real 3×3 matrices A that satisfy $A^T M A = M$, where A^T is the transpose of A.
(a) Show that G is a submanifold of $M_3(\mathbb{R})$, and compute its dimension.
(b) Show that the tangent space to G at the identity matrix I is the set of real 3×3 matrices m with the property that $m^T M + Mm = 0$.

4. Let S^n denote the n-sphere in $\mathbb{R}^{n+1}$. Let $a : S^n \to S^n$ be the antipodal map $a(x) = -x$. Also, let $\deg f$ denote the degree of a smooth map f.
(a) Show that $\deg a = (-1)^{n+1}$.
(b) Show that if $f : S^n \to S^n$ is a smooth map with no fixed points then $\deg f = (-1)^{n+1}$.

5. Let M be a smooth manifold with boundary ∂M. Let $i : \partial M \to M$ be the inclusion. A *retraction* of M to ∂M is a smooth map $r : M \to \partial M$ with the property that $r \circ i$ is the identity map on ∂M.

   Use Stokes' theorem to show that a smooth, orientable, compact manifold with boundary has no retraction to the boundary.

6. Let S^n denote the n-sphere in $\mathbb{R}^{n+1}$. Prove the following statements.
   (a) Any continuous map $S^2 \to S^1 \times S^1$ is homotopic to a constant map.
   (b) There is a continuous map $S^1 \times S^1 \to S^2$ that is not homotopic to a constant map.

7. Show that
   (a) $\mathbb{R}^2$ is homotopy equivalent to $\mathbb{R}^3$, and
   (b) $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.

8. Let X be the space obtained from D^2 by identifying points on the boundary with their images under a $2\pi/3$ rotation.
   (a) Show that $\pi_1(X)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.
   (b) Find a space with fundamental group isomorphic to the free product $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.
   (c) Find a space with fundamental group isomorphic to the direct product $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.