A Conjecture of Solymosi on Lines and Incidences (Part 2)

Chris Pryby
Georgia Institute of Technology

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Theorem (Solymosi's Conjecture)

For every $\varepsilon > 0$ there exists $\delta > 0$ such that, for n sufficiently large, if $\mathcal{L}$ is a collection of n^{ε} GP lines, then not every line in $\mathcal{L}$ is $n^{1-\delta}$-rich.

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- From this idea, construct more lines which are also rich in the grid by composing ℓ_1^{-1} with ℓ_2
- IP($\mathcal{L}$) denotes the set of lines obtained via the intersection-projection procedure:

$$\text{IP}(\mathcal{L}) = \left\{\ell_1^{-1} \circ \ell_2 : \ell_i \in \mathcal{L}\right\}$$

Towards a Proof

Lemma

If $\mathcal{L}$ is a set of K lines which are $n^{1-\delta}$-rich, then for $\approx K^2 n^{-O(\delta)}$ pairs (i,j), $\ell_i^{-1} \circ \ell_j$ is $n^{1-O(\delta)}$-rich

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  - Stop producing many new rich lines with a new iteration

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- Goal: Show that most of our lines are in GP

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  - A large subset of lines in GP

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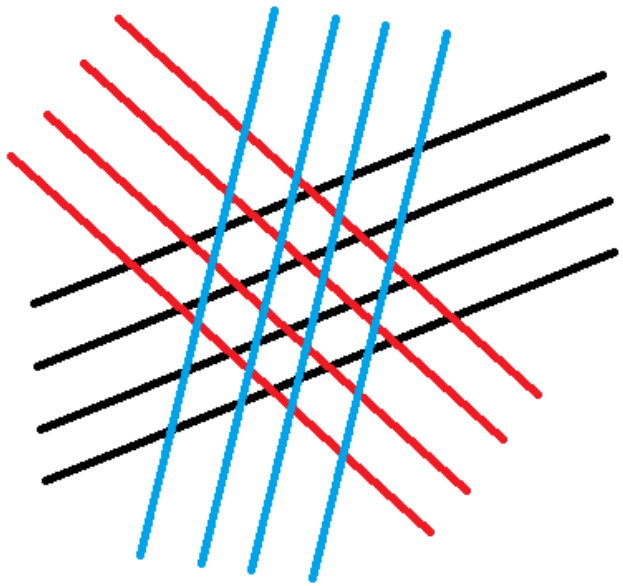
- Similar argument for large star families

Moderate Number of Parallel Families

- $n^{\varepsilon/4}$ families of rich parallel lines after one IP iteration



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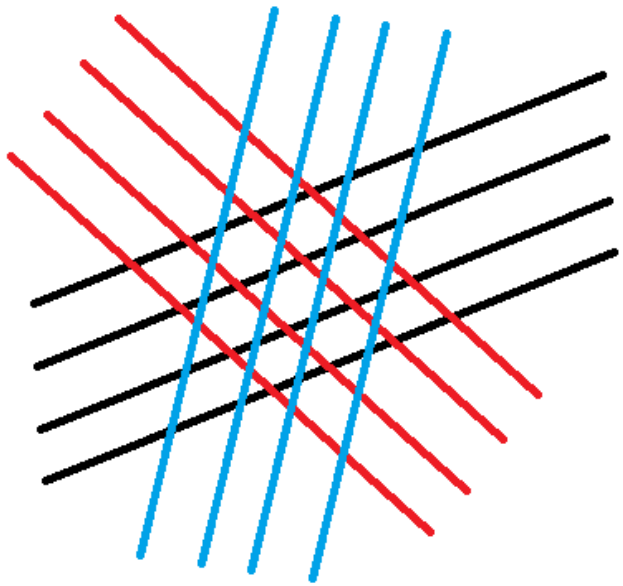

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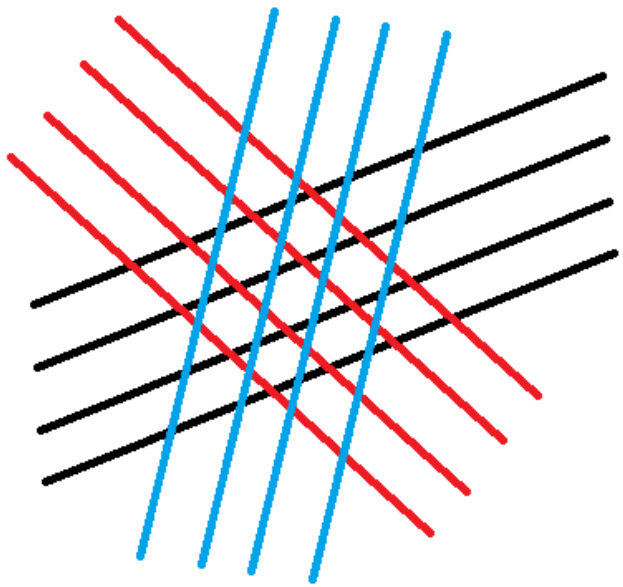

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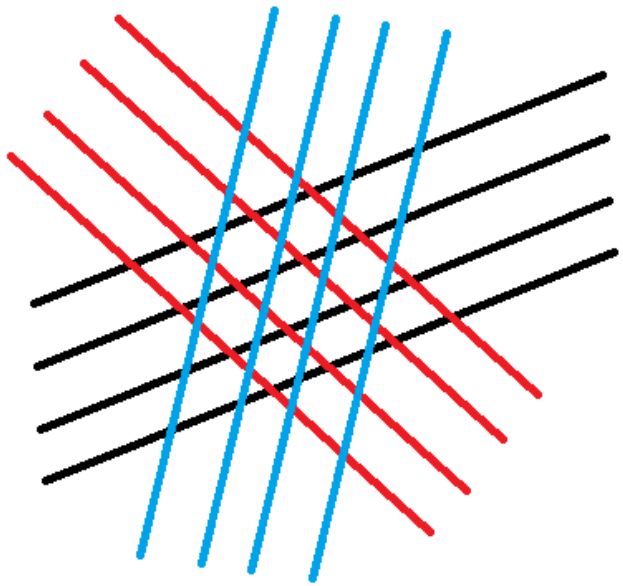

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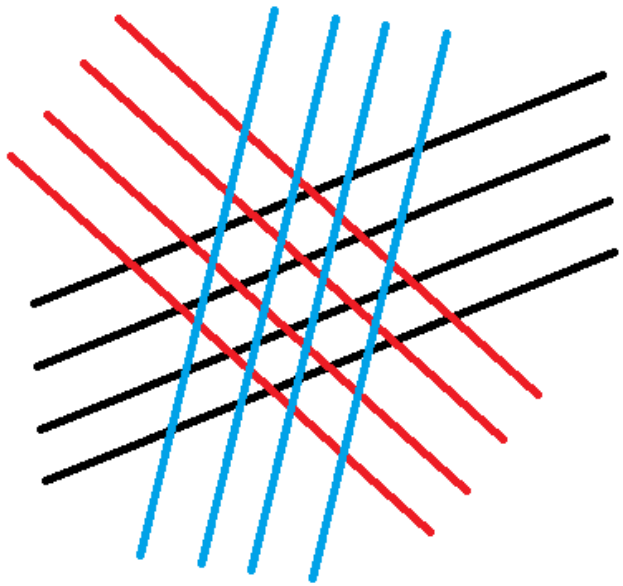

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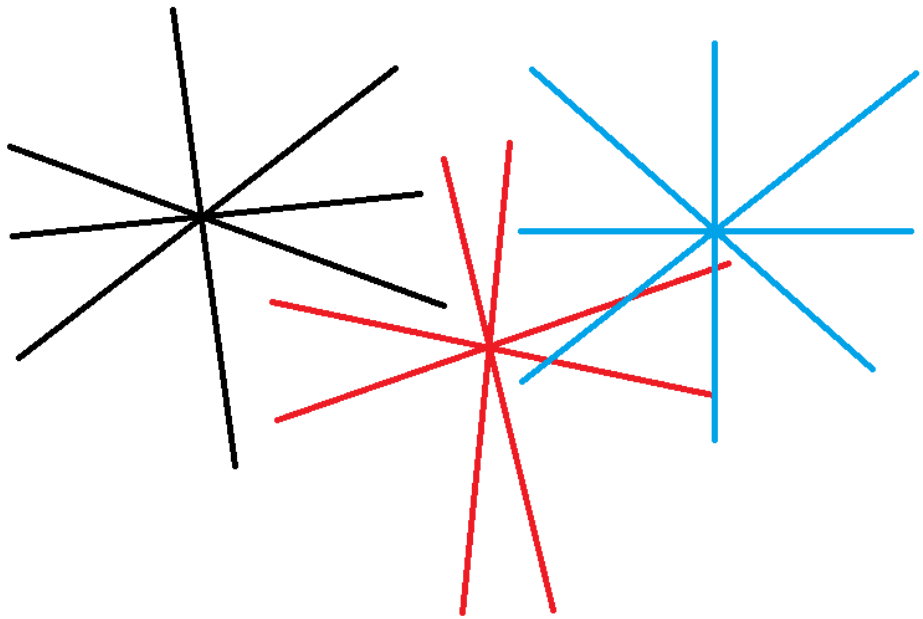

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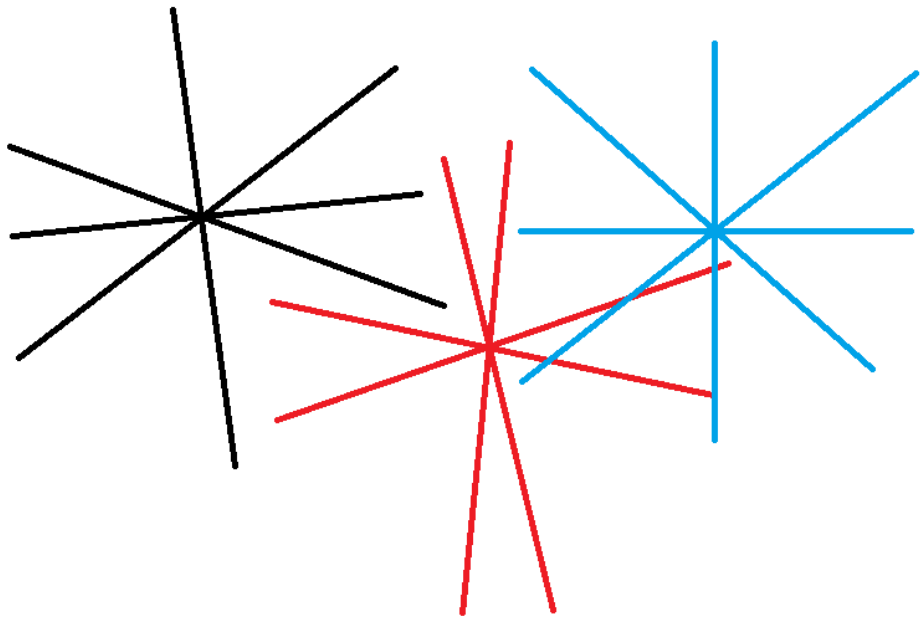

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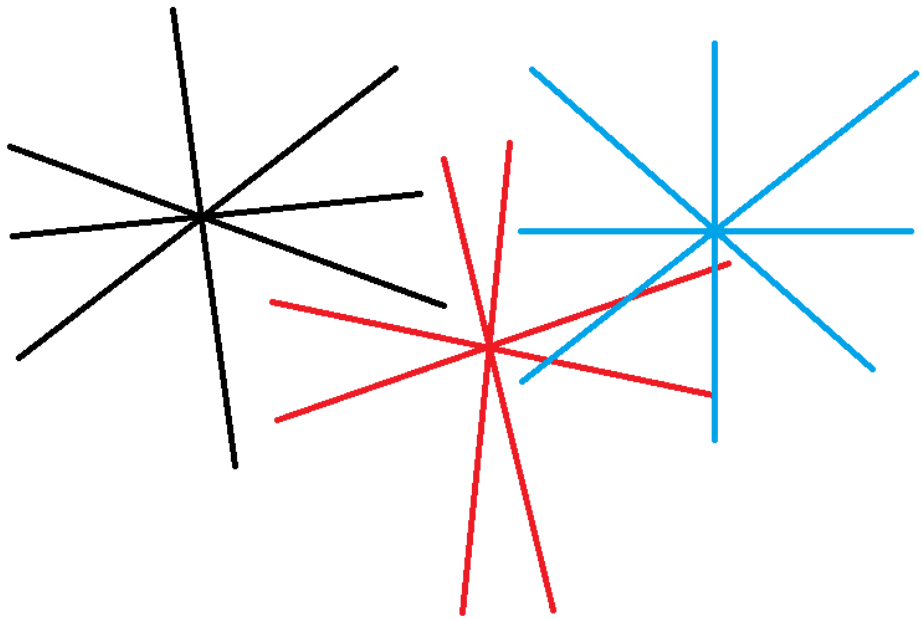

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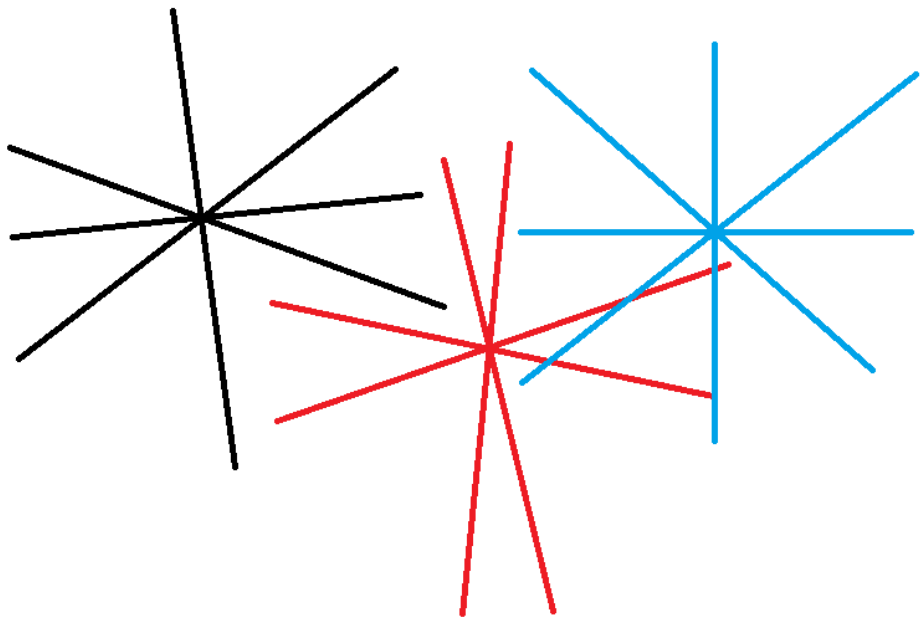

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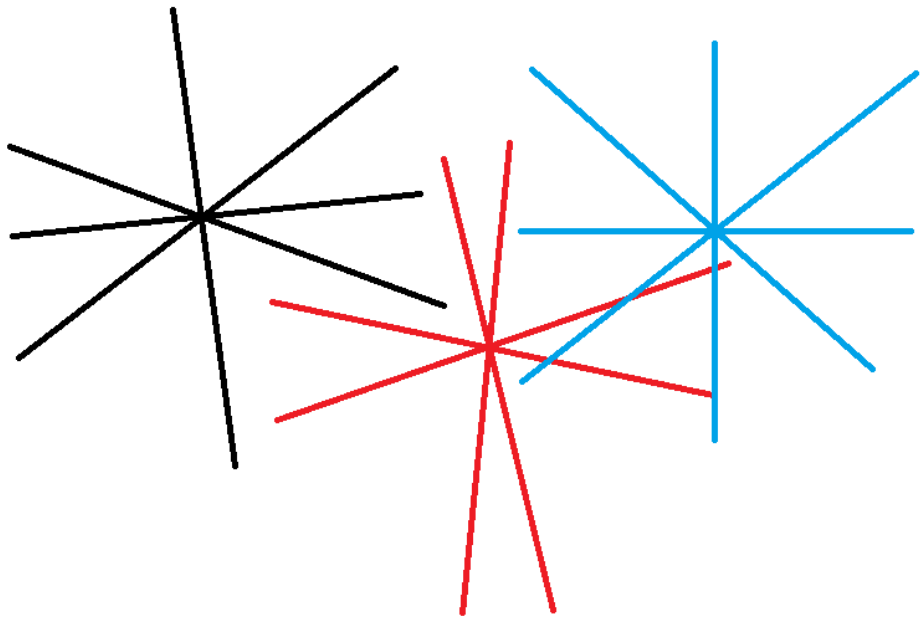

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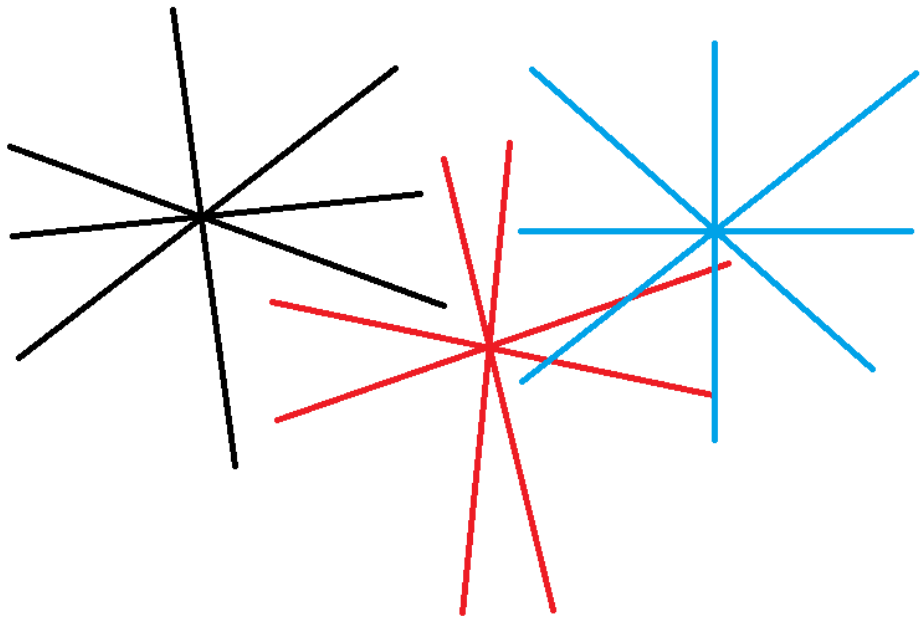
  - Or else we get too many rich lines: contradiction



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For every $0 < \varepsilon < 1/2$ *and* $c > 1$, *there is* $\delta > 0$ *such that for large* k *and large subsets* A *of an additive abelian group: if* $S \subseteq A \times \cdots \times A = A^k$ *has size at least* $|A|^{k-\delta}$ *and* $\Sigma(S) = \{s_1 + \cdots + s_k : (s_1, ..., s_k) \in S\}$ *has size at most* $|A|^c$, *then there is a subset* $A' \subseteq A$ *with size at least* $|A|^{1-\varepsilon}$ *so that* $|\ell A'| \leq |A'|^{c(1+\varepsilon\ell)}$.

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Theorem (Croot-Hart)

For every $c > 0$ there exists $\beta > 0$ and $k \geq 1$ such that for large N, the following holds: if $B \subseteq \mathbb{R}$ has size N and $|B.B| \leq N^{1+\beta}$, then $|kB| \geq N^c$.

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