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Sharp sufficient attractivity conditions for sliding on a co-dimension 2 discontinuity surface

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Abstract

We consider Filippov sliding motion on a co-dimension 2 discontinuity surface. We give conditions under which $\Sigma$ is attractive through sliding which are sharper than those given in a previous paper of ours. Under these sharper conditions, we show that the sliding vector field considered in the same paper is still uniquely defined and varies smoothly in $x \in \Sigma$. A numerical example illustrates our results.

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Keywords: Piecewise smooth systems; Filippov systems; Sliding modes; Discontinuity surface; Co-dimension 2; Attractivity

1. Introduction

An outstanding problem in the study of piecewise smooth differential systems is how to properly define a Filippov sliding vector field when sliding motion has to take place on a co-dimension 2 surface, $\Sigma$, intersection of two co-dimension 1 surfaces. In [3], we gave sufficient conditions which guaranteed that $\Sigma$ attracted nearby trajectories (through sliding), and that a certain sliding vector field, (7) below, was well defined on $\Sigma$. Our goal in this work is to sharpen the conditions given in [3], while still obtaining the same conclusions.

The basic problem we consider is the piecewise smooth system

$$\dot{x} = f(x), \quad f(x) = f_i(x), \quad x \in R_i, \quad i = 1, \ldots, 4,$$

with initial condition $x(0) = x_0 \in R_i$, for some $i$. Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, and (locally) $\mathbb{R}^n = \bigcup_i R_i$. Each $f_i$ can be assumed smooth in an open neighborhood of the closure of each $R_i$, $i = 1, \ldots, 4$. Clearly, from (1), the vector field is not properly defined on the boundaries of the $R_i$'s.

Above, we will assume that the $R_i$'s are (locally) separated by two intersecting smooth surfaces of co-dimension 1, $\Sigma_1 = \{x : h_1(x) = 0\}$ and $\Sigma_2 = \{x : h_2(x) = 0\}$, and we let $\Sigma = \Sigma_1 \cap \Sigma_2$. We will always assume that $\nabla h_1(x) \neq 0, x \in \Sigma_1$.

This work was done while the second and third author were visiting the School of Mathematics of Georgia Institute of Technology, whose hospitality is gratefully acknowledged.

\[ \nabla h_2(x) \neq 0, \ x \in \Sigma, \text{ that } h_{1,2} \text{ are } C^k \text{ functions, with } k \geq 2, \text{ and further that } \nabla h_1(x) \text{ and } \nabla h_2(x) \text{ are linearly independent for } x \text{ on (and in a neighborhood of) } \Sigma. \]

Without loss of generality, we can label the regions as follows:

- \( R_1 : f_1 \) when \( h_1 < 0, h_2 < 0 \),
- \( R_2 : f_2 \) when \( h_1 < 0, h_2 > 0 \),
- \( R_3 : f_3 \) when \( h_1 > 0, h_2 < 0 \),
- \( R_4 : f_4 \) when \( h_1 > 0, h_2 > 0 \),

and we will also adopt the notation \( \Sigma_{1,2}^+ \) and \( \Sigma_{1,2}^- \) to denote the set of points \( x \in \Sigma_{1,2} \) for which we also have \( h_{2,1}(x) > 0 \) or \( h_{2,1}(x) < 0 \). See Fig. 1.

Finally, we let

- \( w_{1,1} = \nabla h^T f_1 \)
- \( w_{1,2} = \nabla h^T f_2 \)
- \( w_{2,1} = \nabla h^T f_3 \)
- \( w_{2,2} = \nabla h^T f_4 \)

which we assume to be well defined in a neighborhood of \( \Sigma \). As it turns out, the signs of the \( w_{ij} \)'s are the key property to monitor.

**Remark 1.** Looking ahead, let us suppose that we are following a solution trajectory on \( \Sigma, x(t) \). In this case, we will need to consider the \( w_{ij} \)'s along this solution trajectory, and can thus think of the \( w_{ij} \)'s as functions of \( t \).

**Remark 2.** The classical Filippov theory (see [6]) is concerned with the case of two regions separated by a surface \( \Sigma \) defined as the 0-set of a smooth scalar valued function \( h \):

\[ \Sigma := \{ x \in \mathbb{R}^n : h(x) = 0 \}, \quad h : \mathbb{R}^n \to \mathbb{R}. \]  

Filippov convexification method allows to define a sliding motion on \( \Sigma \), in particular when \( \Sigma \) attracts nearby trajectories. Filippov proposal is to take a convex combination of \( f_1 \) and \( f_2 \) and impose that the vector field is tangent to \( \Sigma \). That is, take \( f_F := (1-\alpha)f_1 + \alpha f_2 \), with \( \alpha \) chosen so that \( f_F \in T_\Sigma \):

\[ x' = (1-\alpha)f_1 + \alpha f_2, \quad \alpha = \frac{w_1}{w_1 - w_2}, \quad w_1 = \nabla h(x)^T f_1(x), \quad w_2 = \nabla h(x)^T f_2(x) \]  

With the above in mind, we can consider sliding on \( \Sigma_{1,2}^\pm \) previously defined (if such sliding motion indeed can take place). According to (5), we will call \( f_{\Sigma_{1,2}^\pm} \) these four vector fields, defined as follows (as long as the denominators are nonzero):
2. Background

When attempting to define a Filippov sliding vector field on $\Sigma = \Sigma_1 \cap \Sigma_2$, one needs to consider a convex combination of the four vector fields $f_1, \ldots, f_4$. If $f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4$, $\lambda_i \geq 0$, $i = 1, \ldots, 4$, and $\sum \lambda_i = 1$. Imposing that $f \in T_{\Sigma}$, however, is no longer sufficient (unlike the case of Remark 2) to uniquely determine the coefficients $\lambda_i$’s.

To resolve the above ambiguity, in [2,5,1] the authors proposed to restrict consideration to the following bilinear vector field

$$f_F = (1 - \alpha)(1 - \beta)f_1 + (1 - \alpha\beta)f_2 + \alpha(1 - \beta)f_3 + \alpha\beta f_4,$$

where now $\alpha, \beta \in [0, 1]$ need to be found so to satisfy the following nonlinear system:

$$(1 - \alpha)(1 - \beta) \begin{bmatrix} w^1_1 \\ w^2_1 \end{bmatrix} + (1 - \alpha\beta) \begin{bmatrix} w^1_2 \\ w^2_2 \end{bmatrix} + \alpha(1 - \beta) \begin{bmatrix} w^1_3 \\ w^2_3 \end{bmatrix} + \alpha\beta \begin{bmatrix} w^1_4 \\ w^2_4 \end{bmatrix} = 0.$$  

The question then becomes solvability (unique) of this system. To address this problem, in [3] we considered the case of $\Sigma$ being reached through sliding on one of the $\Sigma_{1,2}$, and to characterize this situation we worked under the following assumptions.

**Assumptions 1.**

(a) $(w^+_{j1}(x), w^+_{j2}(x))$ do not have the same signs as $(h_1(x), h_2(x))$ for $x \in R_i$, $j = 1, 2, 3, 4$.

(b) At least one pair of the relations $(1^+)$and$(1^-)$, or $(2^+)$and$(2^-)$, or $(3^+)$and$(3^-)$, or $(4^+)$and$(4^-)$, is satisfied on $\Sigma$ and in a neighborhood of $\Sigma$, where

$$(1^+)w^1_1 > 0, w^1_4 < 0, (1^-)w^1_3 > 0, w^1_3 < 0,$$

$$(2^+)w^2_2 > 0, w^2_4 < 0, (2^-)w^2_3 > 0, w^2_3 < 0,$$

(c) If any of $(1^\pm)$ or $(2^\pm)$ is satisfied, then $(1^\pm_a)$ or $(2^\pm_a)$ must be satisfied as well.

Let us clarify the meaning of Assumptions 1 insofar as the dynamics of the system. Assumption 1(a) implies that the vector fields $f_j$, $j = 1, \ldots, 4$, must point toward at least one of $\Sigma_{1,2}$. Assumption 1(b) guarantees that there is attractive sliding toward $\Sigma$ along at least one of the $\Sigma_{1,2}^\pm$. Assumption 1(c) states that if attractive sliding occurs along $\Sigma_{1,2}^\pm$ it must be toward $\Sigma$. 

It must be emphasized that our theory is justified under the assumption that $\Sigma$ is attractive in finite time upon sliding on a co-dimension 1 surface. Hence, Assumption 1(c) are fundamental in this setting.

In [3], we made a simplifying assumption on the $w_j$’s, expressed by the following:

**Old assumption (see [3]):**

$$\frac{w_j(x)}{w_4(x)} \text{ are bounded away from } 0, \quad i = 1, 2, 3, 4, \quad x \in \Sigma.$$  \hspace{1cm} (9)

Note that (9) implies that no trajectory can approach $\Sigma$ tangentially from a region $R_j, j = 1, 2, 3, 4$.

Under Assumptions 1 and 2, in [3] it was proved that $\Sigma$ attracted nearby trajectories, which in fact reached $\Sigma$ in finite time, and moreover that (8) had a unique solution. More precisely, we proved the following result.

**Theorem 3.** Let Assumptions 1 be satisfied and let (9) hold.

(a) Then, there exists a unique solution $(\bar{x}, \bar{\beta})$ of system (8) in $(0, 1) \times (0, 1)$.

(b) Further, let $(1_1^\pm), (1_2^\pm), (2_2^\pm)$, and $(2_4^\pm)$, hold uniformly; that is $(1_1^\pm)$ be replaced by $(w_2^2/w_4^2) - (w_4^2/w_4^4) \leq -\lambda_1^+ < 0$, and similarly for the others. Then, $\Sigma$ is attractive in finite time.

3. Weaker attractivity assumptions

It was already observed in [3] that (9) was too strong a sufficient condition to guarantee the conclusions of Theorem 3. For this reason, our goal below is to weaken (9) in such a way that: $\Sigma$ still remains attractive through sliding and reached in finite time, and the vector field (7) still is well defined on $\Sigma$.

We restrict ourselves to co-dimension 1 phenomena, as characterized by having just one scalar value among the $w_j$’s being 0 at any point in $\Sigma$. Higher co-dimension phenomena (such as two of the $w_j$’s becoming 0 at the same time) are not necessarily going to preclude the aforementioned conclusions (i.e., attractivity of $\Sigma$ and well posedness of the vector field (7)), but require a host of different possibilities to be examined, which is beyond our present scope.

**Assumptions 2.** At most one of the $w_j$’s is zero at any given $x$ on $\Sigma$.

In this paper we replace condition (9) with Assumptions 2. This means that, while sliding on $\Sigma$, one of the $w_j$’s can be zero at a point $x \in \Sigma$, as long as Assumptions 1 are still satisfied.

Assumptions 1 and 2 together imply the following:

(i) $f_j$ cannot be tangential to $\Sigma$ at $x \in \Sigma$;

(ii) $f_j$ cannot be tangential to $\Sigma_2$ (respectively, $\Sigma_1$), at a point $x$ on $\Sigma$, whenever $f_j$ points away from $\Sigma_1$ (respectively, $\Sigma_2$) at $x$.

Item (i) above is just a rewriting of Assumptions 2. To exemplify the instances in (ii), assume that $\Sigma$ is attractive and that we are following a trajectory on $\Sigma$. Assumptions 1 are satisfied along the trajectory and take for example $w_1(x(t)) < 0$ and $w_1(x(t)) > 0$ for $t < T$. At $x = x(T), w_1(x) = 0$ while $w_1(x)$ stays negative. Now $f_j$ is tangent to $\Sigma_2$, but points away from $\Sigma_1$. Thus, for $t > T, w_2(x(t)) < 0$ and $w_1(x(t)) < 0$. This, together with the continuity of $f_1$, violates Assumptions 1(a). The vector $f_1$ now points away from $\Sigma$ so that $\Sigma$ loses attractivity. The same reasoning as above applies to any other vector $w_j = (w_j^1, w_j^2)$.

**Fig. 2** shows the admissible configurations for $f_1$ at $x \in \Sigma$. Here we take $w_1(x) < 0$ and we show only the component of $f_1$ in the normal plane at $x$ to $\Sigma$. The dotted and dashed vectors are not admissible due to Assumptions 1(a), while...
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Table 1

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<thead>
<tr>
<th>Component</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{1i}^j$</td>
<td>$\leq 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>$w_{2i}^j$</td>
<td>$&gt; 0$</td>
<td>$\geq 0$</td>
<td>$\leq 0$</td>
<td>$&lt; 0$</td>
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Table 2

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<th>Component</th>
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<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
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<tbody>
<tr>
<td>$w_{1i}^j$</td>
<td>$\leq 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>$w_{2i}^j$</td>
<td>$&gt; 0$</td>
<td>$\geq 0$</td>
<td>$\geq 0$</td>
<td>$&lt; 0$</td>
</tr>
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</table>

the solid vectors are admissible configurations. The dashed vector has first component $w_{1i}^j < 0$ and second component $w_{2i}^j = 0$.

On the other hand, if, as before, while sliding on $\Sigma$, $w_{1i}^j(x) = 0$, but $w_{1i}^j(x) > 0$ along the trajectory, Assumptions 1(a) are not violated, (see the numerical example in Section 4 where $x = x_5$) and, as we will show in Lemma 4, $\Sigma$ retains attractivity in finite time. This shows how Assumptions 1(a) are sharper than condition (9).

**Lemma 4.** Let Assumptions 1 and 2 be verified and let (1$^\pm$) and (2$^\pm$) hold uniformly, then $\Sigma$ is attractive in finite time.

**Proof.** The proof is analogous to the proof of Lemma 4 in [3]. We only outline the first part of the proof since it is slightly different. Assumptions 1(a) and 2 guarantee that every $f_j$, $j = 1, 2, 3, 4$ points toward at least one of $\Sigma_1$ or $\Sigma_2$. This, together with the fact that $f_j$ is never tangent to $\Sigma$, guarantees that if we start in $R_j$, we reach $\Sigma_1$ or $\Sigma_2$ or $\Sigma$ in finite time. The rest of the proof is the same as that of [3, Lemma 4].

It must be emphasized that a sign change of the $w_{1i}^j$’s, while Assumptions 1 are still satisfied, does not necessarily lead to a loss of attractivity of $\Sigma$.

**Remark 5.** Having one of the $f_j$’s tangent to $\Sigma$ at a point $\bar{x} \in \Sigma$, is neither a necessary nor a sufficient condition for loss of attractivity of $\Sigma$. Hence, if, while sliding on $\Sigma$, $w_{1i}^j(\bar{x}) = w_{1i}^j(\bar{x}) = 0$, this does not necessarily mean that the trajectory should exit $\Sigma$ tangentially with vector field $f_j$, as one may expect according to a first order theory. Contrast this to the case of sliding on a co-dimension 1 surface where, if one of the vector fields is tangent to $\Sigma$ at a point $\bar{x}$, there is loss of attractivity of $\Sigma$ and (according to a first order theory) a tangential exit from $\Sigma$.

The theorem below shows that Assumptions 1 and 2 are sufficient for (7) to be well defined on $\Sigma$.

**Theorem 6.** Let Assumptions 1 and 2 hold. Then, there exists a unique solution $(\bar{a}, \bar{b})$ of system (8) in $(0, 1) \times (0, 1)$.

**Remark 7.** Assumptions 1 and 2 are not necessary to have a unique solution $(\bar{a}, \bar{b})$ of (8) on $(0, 1)^2$. As a matter of facts, the vector field (7) might exist and be unique even if $\Sigma$ is not attractive (see Example 10).

**Proof.** The proof is analogous to the proof of Theorem 3 in [3], and below we will highlight just those cases requiring modifications to the arguments used in [3].

In what follows we consider some of the cases that appear in the proof of Theorem 3 in [3], and use the labeling of these cases using the same notation adopted in [3]. These cases are chosen so that each one occurs from one of the others due to a sign change of one of the $w_{1i}^j$’s. In this way, we can visualize the change in dynamics around $\Sigma$ if one of the $w_{1i}^j$’s changes sign, with Assumptions 1 and 2 holding. We emphasize that under each of these sign changes $\Sigma$ retains attractivity in finite time. In all the tables below (Tables 1–4) we show the sign of each component $w_{1i}^j$ in these tables, the writing $\geq 0$, $\leq 0$, must be understood within the limitations imposed by Assumption 2: at any given point $\bar{x} \in \Sigma$ only one of the $w_{1i}^j$’s is allowed to be zero. In the corresponding figures (Figs. 3–6) we only display the $w_{1i}^j$’s when they are all different from zero.

Table 3

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<tr>
<th>Component</th>
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<th>$i = 2$</th>
<th>$i = 3$</th>
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</thead>
<tbody>
<tr>
<td>$\mu_1^i$</td>
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<td>$&gt; 0$</td>
<td>$\geq 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>$\lambda^i$</td>
<td>$&gt; 0$</td>
<td>$\geq 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
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Table 4

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<th>Component</th>
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<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1^i$</td>
<td>$\geq 0$</td>
<td>$&gt; 0$</td>
<td>$\leq 0$</td>
<td>$\leq 0$</td>
</tr>
<tr>
<td>$\lambda^i$</td>
<td>$\geq 0$</td>
<td>$\geq 0$</td>
<td>$\geq 0$</td>
<td>$&lt; 0$</td>
</tr>
</tbody>
</table>

Case ($S_{\Sigma^+_1} : 2$) The signs of the entries of $w_1^i$ and $w_2^i$ are as in Table 1, and the following condition is satisfied

$$\frac{w_2^1}{w_2^i} < \frac{w_2^2}{w_2^4}$$

(10)

Condition (10) ensures sliding along $\Sigma^+_1$ toward $\Sigma$.

According to Assumptions 1, $w_1^i$ can be zero on $\Sigma$ since $w_1^2 > 0$, and similarly for $w_2^3$ and $w_3^4$. Notice, instead, that $w_1^1$ must be bounded away from zero even though $w_4^2 < 0$. This is in order to ensure sliding on at least one of the co-dimension 1 surfaces. Indeed assume that, while following...
a trajectory on $\Sigma$, at $t = T$, $w_4(x(T)) = 0$. Then for $t > T$ and $t$ sufficiently close to $T$, $w_4(x(t)) > 0$ and there is no sliding on a co-dimension 1 surface. This is against Assumptions 1(b).

Case ($S_{\Sigma_1^+,\Sigma_2^+} : 2$) This case follows from Case ($S_{\Sigma_1^+,\Sigma_2^+} : 2$) above, here $w_3$ has undergone a sign change.

The signs of the entries of $w_1$ and $w_2$ are as in Table 2 and (10) is satisfied. See Fig. 4.

Case ($S_{\Sigma_1^+,\Sigma_2^-} : 5$) This case follows from Case ($S_{\Sigma_1^+,\Sigma_2^-} : 2$) above, here $w_3$ has undergone a sign change.

The signs of the entries of $w_1$ and $w_2$ are as in Table 3 and condition (10) is satisfied together with the following:

$$\frac{w_1}{w_2} < \frac{w_3}{w_4}$$

(11)

see Fig. 5.

Here ($1_a^+$) and ($2_a^+$) imply that $w_4^l$ and $w_4^r$ must be different from zero.

Case ($S_{\Sigma_1^+,\Sigma_2^-} : 2$) This case follows from Case $S_{\Sigma_1^+,\Sigma_2^-} : 2$ above after $w_4^l$ has undergone a sign change.

The signs of the entries of $w_1$ and $w_2$ are given in Table 4 and (11) is satisfied; see Fig. 6.

For this configuration ($1_a^+$) implies that $w_4^l$ must be different from zero.

**Example 8.** Here we illustrate all the changes in dynamics that might occur while sliding on $\Sigma$ under Case ($S_{\Sigma_1^+,\Sigma_2^-} : 2$), when one of the components allowed to be zero in Table 2 goes to zero. Suppose that, while following a trajectory
Table 5

<table>
<thead>
<tr>
<th>Component</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1^j$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-0.5</td>
</tr>
<tr>
<td>$w_2^j$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Figure 7. Example 10, $w_j^i$'s at $t = T$.

$x(t)$ on $\Sigma$, one of the $w_j^i$'s is zero at time $t = T$. We will list here all the possible changes in dynamic that occur after time $T$.

1. If $w_1^1(T) = 0$, then the dynamic after time $T$ is the one in Case $(S_{\Sigma^+_{1,2}}, \Sigma^+_{2}) : 2$.
2. If $w_1^3(T) = 0$, then the dynamic after time $T$ is the one in Case $(S_{\Sigma^+_{1}}, \Sigma^+_{2}) : 2$.
3. If $w_2^3(T) = 0$, then the dynamic after time $T$ is the one in Case $(S_{\Sigma^+_{1}}, \Sigma^+_{2}) : 2$.
4. If $w_1^4(T) = 0$, then after time $T$ there is attractive sliding only along $\Sigma^+_{2}$ and there is no sliding along $\Sigma^+_{1}$.
5. If $w_2^2(T) = 0$, then after time $T$ there is attractive sliding toward $\Sigma$ along $\Sigma^+_{1}$ and $\Sigma^+_{2}$. This case mirrors Case $(S_{\Sigma^+_{1,2}}, \Sigma^+_{2}) : 2$.

We stress that $\Sigma$ is attractive in a neighborhood of $x(T)$, and that (7) is well defined.

Just like in [3, Theorem 8], and with the same proof, the following holds.

Theorem 9. Under Assumptions 1 and 2, the unique solution $(\alpha, \beta) \in (0, 1) \times (0, 1)$ of system (8) varies smoothly with respect to $x \in \Sigma$.

3.1. Loss of attractivity

In [3] we showed that violating any of (1$^\pm$) or (2$^\pm$) in Assumptions 1 leads to a loss of attractivity of $\Sigma$. We further identified when/how this loss of attractivity condued to an exit from $\Sigma$ to slide on one of $\Sigma_{1,2}$, first order exit condition.

Now, when only one of the $w_j^i$'s is zero, and Assumptions 1 are not satisfied, then $\Sigma$ looses attractivity, but the vector field (7) might still be defined on $\Sigma$ as showed in Example 10.

Example 10. Assume that we are following a trajectory on $\Sigma$ with the $w_j^i$'s as in Table 1. Moreover, assume that all the $w_j^i$'s are bounded away from zero for $t < T$ and that at time $t = T$ they are as in Table 5 and Fig. 7. As it is clear from Fig. 7, at $x(T)$ Assumptions 1 are not satisfied and $\Sigma$ loses attractivity at time $t = T$. Nonetheless, system 8 still admits a unique solution $(\alpha, \beta) \simeq (0.4226, 0.7321)$, hence the vector field (7) is still well defined on $\Sigma$. 

In Remark 5, we noticed how, while sliding on $\Sigma$, $f_j$ might be tangent to $\Sigma$ without this implying a loss of attractivity of $\Sigma$. Here, in Remark 11, we emphasize how a co-dimension 2 sliding surface might lose attractivity at a point $x$ even though there is no potential tangential exit vector field at that point.

**Remark 11.** Consider again Table 5. Note that $\Sigma$ has lost attractivity, but there is no tangential vector field exiting $\Sigma$. This is in distinct contrast with sliding on a co-dimension 1 surface. In the latter case, indeed, when the sliding surface $\Sigma$ loses attractivity, Filippov theory will predict (at first order) exiting $\Sigma$ tangentially.

### 4. Numerical example

Here we result of numerical experiments on an example where one of the $w_j$'s (namely, $w_1^2$) becomes 0 along the sliding trajectory, still satisfying Assumptions 1. Aside from the modification due to $w_1^2$ becoming 0, the example below is actually the one we meant to use in [3].

All computations have been made with an event driven technique, and event points (when a different regime is reached) have been computed by the secant method. Integration of all relevant differential equations was made using the classical explicit Runge-Kutta (RK) scheme of order four, a projected RK method in case of sliding motion to ensure that all evaluations are made on the constraints’ surfaces (e.g., see [4]). The stepsize $\tau$ was held constant and equal to $\tau = 0.0025$, and of course adjusted when using the secant method to locate event points. Solution of the system (8) was done by Newton’s method.

**Example 12.** We have the discontinuity surfaces

$$
\Sigma_1 = \{x \in \mathbb{R}^3 : h_1(x) = x_2 - p\}, \quad \Sigma_2 = \{x \in \mathbb{R}^3 : h_2(x) = x_3 - q\}, \quad \Sigma = \Sigma_1 \cap \Sigma_2
$$

with $p = 0.5$ and $q = 1$. Thus, we have the following four vector fields, at least continuous in their respective regions of definition:

$$
R_1(h_1 < 0, h_2 < 0) : f_1(x) = \begin{pmatrix} x_2 \\ -x_1 + \frac{1}{(1+p) - x_2} \\ -x_1 + \frac{32}{1 + (1+q) - x_3} \end{pmatrix}
$$

$$
R_2(h_1 < 0, h_2 > 0) : f_2(x) = \begin{pmatrix} -(x_2 + x_3) \\ -x_1 + \frac{1}{(1+p) - x_2} \\ -x_1 - \frac{1}{(1-q) + x_3} \end{pmatrix}
$$

$$
R_3(h_1 > 0, h_2 < 0) : f_3(x) = \begin{cases} 
-6 + 1.3 + \frac{x_1}{1.3} + \frac{1}{(1-p) + x_2} \\ -x_1 + \frac{1}{1 + (1+q) - x_3} \end{cases}
$$

when $x_1 \geq -1.3$,

$$
R_4(h_1 > 0, h_2 > 0) : f_4(x) = \begin{cases} 
-6 + 1.3 + \frac{x_1}{1.3} + \frac{1}{(1-p) + x_2} \\ -x_1 + \frac{1}{1 + (1+q) - x_3} \end{cases}
$$

when $x_1 < -1.3$.
Fig. 8. Solution trajectory: the solution spirals around $\Sigma$, starts sliding on $\Sigma^+_1$, enters $\Sigma$ and leaves it to slide on $\Sigma^+_2$.

Fig. 9. Solution trajectory: the solution slides on $\Sigma^+_1$ and leaves it to enter $R_4$, hits $\Sigma^+_2$ and starts sliding on it, then slides $\Sigma$ and starts sliding on it.

$$R_{4}(h_1 > 0, h_2 > 0) : f_4(x) = \begin{cases} 
-(x_2 + x_3) \\
-x_1 - \frac{1}{(1 - p) + x_2} \\
-x_1 + \frac{1}{(1 - q) + x_3} \\
-(x_2 + x_3) \\
130 + 129x_1 + \frac{1}{(1 - q) + x_3}
\end{cases} \quad \text{when } x_1 \geq -1,
$$

$$\begin{cases} 
-x_1 - \frac{1}{(1 - p) + x_2} \\
-x_1 + \frac{1}{(1 - q) + x_3} \\
-(x_2 + x_3) \\
130 + 129x_1 + \frac{1}{(1 - q) + x_3}
\end{cases} \quad \text{when } x_1 < -1.$$

Results below are for initial condition $x_0 = [0.7, 0.49, 0.99]$. We can distinguish several different dynamics of the solution with respect to the two discontinuity surfaces. Indeed, there are several event points, that is values where the solution reaches a different regime: a different region and/or sliding surface. We will assign a time value $t_j$, and with abuse of notation indicate each event point with $x_j$. The initial part of the trajectory is plotted in Fig. 8, and Fig. 9 shows the entire trajectory; event points are marked by asterisks: $x_j, j = 1, \ldots, 11$. 

The initial condition is in region $R_1$ and the trajectory crosses $\Sigma_1^-$ at $x_1 \approx (0.71728, 0.5, 0.98318)$ and enters $R_3$ (transversal intersection). At $x_2 \approx (0.63696, 0.51686, 1)$, it crosses $\Sigma_2^+$ and enters $R_4$ (transversal intersection). At $x_3 \approx (0.62125, 0.5, 1.00384)$, it hits $\Sigma_1^+$ and starts sliding on it in the direction of $\Sigma$ with vector field $f_{\Sigma_1^+}$. Then, while sliding on $\Sigma_1^+$ the solution reaches $\Sigma$ at time $t_4$ at the point $x_4 \approx (0.61659, 0.5, 1)$. At $x_4$, the vector fields $f_j, j = 1, 2, 3, 4$, have the signs given in Table 6 and condition (10) is satisfied, so that Assumptions 1 are satisfied, $\Sigma$ is attractive, $f_\Sigma$ as in (7) is well defined and the solution starts sliding on $\Sigma$. At time $t_5$, the solution is at $x_5 = (0.5, 0.5, 1)$, $w_1(x_5) = 0$, and $w_1(x) = 0$ for values on $\Sigma$ in a neighborhood of $x_5$. Assumptions 1 and 2 are satisfied at $x_5$, so the solution keeps sliding on $\Sigma$. The configuration along the solution path in a neighborhood of $t_5$ is the one showed in Fig. 10 (it mirrors Case $(\Sigma_1^+, \Sigma_2^+ : 1)$ in [3]). Here the dashed vector is $w_1 = (w_1^1, w_1^2)$ at a specific time $t < t_5$, the dotted vector is $w_1(x(t_5))$ and the solid vector is $w_1$ at a specific time $t > t_5$. At $t = t_6 \approx 0.62925$, the solution is at $x_6 = (0, 0.5, 1)$, and there is equality in (10). Moreover, the $w^i_j(x)$'s for $x = x_6$ are as in Table 7 and the trajectory leaves $\Sigma$ smoothly to enter $\Sigma_1^+$. So, at $t = t_6$, $f_{\Sigma_1^+}$ aligns to $f_{\Sigma_1^+}$ and the solution exits $\Sigma$ smoothly to slide on $\Sigma_1^+$. At time $t_7 \approx 1.22698$, the solution reaches $x_7 \approx (-1, 0.5, 1.5111)$, $f_{\Sigma_1^+}$ aligns to $f_4$ and the solution exits $\Sigma_1^+$ smoothly to enter in region $R_4$. At time $t_8$, it reaches $\Sigma_2^+$ at $x_8 \approx (-1.322, 0.5050, 1)$, and here $w_2^3 > 0$ while $w_2^3 < 0$ so that sliding begins on $\Sigma_2^+$ away from $\Sigma_1$. At time $t_9 \approx 1.40259$, the solution reaches the surface $x_1 = -1.3$ at $x_9 \approx (-1.3, 0.7105, 1)$; here, $f_3$ is continuous.

### Table 6

<table>
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<tr>
<th>Component</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
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<tbody>
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<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td>$&lt;0$</td>
</tr>
<tr>
<td>$w_2^j$</td>
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<td>$&lt;0$</td>
<td>$&gt;0$</td>
<td>$&gt;0$</td>
</tr>
</tbody>
</table>

### Table 7

<table>
<thead>
<tr>
<th>Component</th>
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<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1^j$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$w_2^j$</td>
<td>$\frac{1}{2}$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

but not differentiable. At time $t_{10}$, we reach the value $x_{10} \approx (-1.7377, 0.9507, 1)$. For $t > t_{10}$ the trajectory continues sliding on $\Sigma_2^+$ but now in the direction of $\Sigma_1$, since the following condition is satisfied:

$$
\frac{w_1^1}{w_2^1} < \frac{w_1^2}{w_2^2}.
$$

At time $t_{11}$, the solution reaches the point $x_{11} \approx (-2.3430, 0.5, 1)$ on $\Sigma$. The vector fields $f_j(x_{10}), j = 1, \ldots, 4$, satisfy the conditions of Table 8 and the behavior on $\Sigma$ is analogous to the one of Case $(S\Sigma_1+ : 2)$. The solution now starts sliding on $\Sigma$ with vector field $f_\Sigma$ as in (7), and remains on $\Sigma$.

### 5. Conclusions

In this paper we weakened the assumptions given in [3] for attractivity of a sliding co-dimension 2 surface $\Sigma$ and for the existence and uniqueness of the Filippov sliding vector field (7) on $\Sigma$. We reported on a numerical experiment to show the behavior of a piecewise smooth system that satisfies our new assumptions.

### References


