Numerical Solution of Discontinuous Differential Systems: Approaching the Discontinuity Surface from One Side

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Abstract

We consider the numerical integration of discontinuous differential systems of ODEs of the type:

\[ x' = f_1(x) \text{ when } h(x) < 0 \] \quad \text{and} \quad \[ x' = f_2(x) \text{ when } h(x) > 0, \]

and with \( f_1 \neq f_2 \) for \( x \in \Sigma \), where \( \Sigma := \{ x : h(x) = 0 \} \) is a smooth co-dimension one discontinuity surface. Often, \( f_1 \) and \( f_2 \) are defined on the whole space, but there are applications where \( f_1 \) is not defined above \( \Sigma \) and \( f_2 \) is not defined below \( \Sigma \). For this reason, we consider explicit Runge-Kutta methods which do not evaluate \( f_1 \) above \( \Sigma \) (respectively, \( f_2 \) below \( \Sigma \)). We exemplify our approach with subdiagonal explicit Runge-Kutta methods of order up to 4. We restrict attention only to integration up to the point where a trajectory reaches \( \Sigma \).

Key words: Discontinuous ODEs, Filippov convexification, Runge-Kutta methods.

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1. Introduction

We study a differential system in \( \mathbb{R}^n \) with discontinuous right hand side in the form:

\[ x'(t) = f(x) = \begin{cases} f_1(x(t)) & \text{when } x \in R_1, \\ f_2(x(t)) & \text{when } x \in R_2, \end{cases} \]  \quad (1)

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The state space $\mathbb{R}^n$ is split (locally) into two subspaces $R_1$ and $R_2$ by a hypersurface $\Sigma$ such that $\mathbb{R}^n = R_1 \cup \Sigma \cup R_2$. The hypersurface is defined by a scalar event function $h : \mathbb{R}^n \to \mathbb{R}$, so that the subspaces $R_1$ and $R_2$, and $\Sigma$, are implicitly characterized as

$$\Sigma = \{x \in \mathbb{R}^n | h(x) = 0\}, \quad R_1 = \{x \in \mathbb{R}^n | h(x) < 0\}, \quad R_2 = \{x \in \mathbb{R}^n | h(x) > 0\}.$$  

We will assume that $h \in C^k$, $k \geq 2$, and that $\nabla h(x) \neq 0$ for all $x \in \Sigma$, so that the unit normal $n$ to $\Sigma$ is well defined\(^1\).

In the differential system (1), the right-hand side $f(x)$ can be assumed to be smooth in $R_1$ and $R_2$ separately, but it is usually discontinuous across $\Sigma$, that is $f_1(x) \neq f_2(x)$, $x \in \Sigma$.

**Remark 1.1.** Although in the literature it is often assumed (e.g. see [2, 4, 3]) that $f_1$ and $f_2$ are defined smoothly everywhere in an open neighborhood of $\Sigma$, this may be a strong restriction from both theoretical (see [2]) and practical points of view. Therefore, we wish to study numerical methods which avoid this assumption.

As we just noticed, in (1) $f(x)$ is not defined if $x$ is on $\Sigma$. A way to define the vector field on $\Sigma$ is to consider the set valued extension $F(x)$ below:

$$x'(t) \in F(x(t)) = \begin{cases} f_1(x(t)), & x \in R_1 \\ \overline{\omega} \{f_1(x(t), f_2(x(t)) \}, & x \in \Sigma \\ f_2(x(t)), & x \in R_2 \end{cases}, \quad (3)$$

\(^1\)In many practical applications, the function $h$ is actually linear ($\Sigma$ is a plane).
where \( \overline{co}(A) \) denotes the smallest closed convex set containing \( A \). In our particular case:

\[
\overline{co}\{f_1, f_2\} = \{f_\alpha : x \in \mathbb{R}^n \to \mathbb{R}^n : f_\alpha = (1 - \alpha)f_1 + \alpha f_2, \ \alpha \in [0, 1]\}.
\] (4)

The extension of a discontinuous system (1) into a convex differential inclusion (3) is known as Filippov convexification. Existence of solutions of (3) can be guaranteed with the notion of upper semi-continuity of set-valued functions ([1], [2]). A solution in the sense of Filippov (see [2]) is an absolutely continuous function \( x : [0, \tau] \to \mathbb{R}^n \) such that \( x'(t) \in F(x(t)) \) for almost all \( t \in [0, \tau] \), where \( F(x(t)) \) is the closed convex hull defined in (4).

Now, consider a trajectory of (1), and suppose that \( x_0 \not\in \Sigma \), and thus, without loss of generality, we can think that \( x_0 \in R_1 \), that is \( h(x_0) < 0 \). The interesting case is when, starting with \( x_0 \), the trajectory of the differential system \( x' = f_1(x) \) is directed towards \( \Sigma \) and reaches it in a finite time. At this point, one must decide what happens next. Loosely speaking, there are two possibilities: (a) we leave \( \Sigma \) and enter into \( R_2 \) (or, less likely, we re-enter in \( R_1 \)); (b) we remain in \( \Sigma \) with a yet to be defined vector field. Filippov devised a very powerful first order theory which helps to decide what to do in this situation, and how to define the vector field in case (b). We summarize it below.

Let \( x \in \Sigma \) and let \( n(x) \) be the normal to \( \Sigma \) at \( x \). Let \( n^T(x)f_1(x) \) and \( n^T(x)f_2(x) \) be the projections of \( f_1(x) \) and \( f_2(x) \) onto the normal direction.

**Transversal Intersection.** In case in which, at \( x \in \Sigma \), we have

\[
[n^T(x)f_1(x)] \cdot [n^T(x)f_2(x)] > 0 ,
\] (5)

then we will leave \( \Sigma \). We will enter \( R_1 \), when \( n^T(x)f_1(x) < 0 \), and will enter \( R_2 \), when \( n^T(x)f_1(x) > 0 \). In the former case we will have (1) with \( f = f_1 \), in the latter case with \( f = f_2 \). Any solution of (1) with initial condition not in \( \Sigma \), reaching \( \Sigma \) at a time \( t_1 \), and having a transversal intersection there, exists and is unique.

**Sliding Mode.** In case in which, at \( x \in \Sigma \), we have

\[
[n^T(x)f_1(x)] \cdot [n^T(x)f_2(x)] < 0 ,
\] (6)

then we have a so-called sliding mode through \( x \).

An attracting Sliding Mode occurs if

\[
[n^T(x)f_1(x)] > 0 \quad \text{and} \quad [n^T(x)f_2(x)] < 0 , \quad x \in \Sigma ,
\] (7)
where the inequality signs depend of course on the definition of $R_{1,2}$ in (2). When we have (7) satisfied at $x_0 \in \Sigma$, a solution trajectory which reaches $x_0$ does not leave $\Sigma$, and will therefore have to move along $\Sigma$: sliding motion. During the sliding motion the solution will continue along $\Sigma$ with time derivative $f_\beta$ given by:

$$f_\beta(x) = (1 - \alpha(x))f_1(x) + \alpha(x)f_2(x).$$

(8)

Here, $\alpha(x)$ is the value for which $f_\beta(x)$ lies in the tangent plane $T_x h(x)$ at $x$, that is the value for which $n^T(x)f_\beta(x) = 0$. This gives

$$\alpha(x) = \frac{n^T(x)f_1(x)}{n^T(x)(f_1(x) - f_2(x))}.$$  

(9)

Observe that a solution having an attracting sliding mode exists and is unique, in forward time.

We have a repulsive sliding mode when

$$[n^T(x)f_1(x)] < 0 \quad \text{and} \quad [n^T(x)f_2(x)] > 0, \quad x \in \Sigma.$$  

(10)

Repulsive sliding modes do not lead to uniqueness (at any instant of time one may leave with $f_1$ or $f_2$), and we will not further consider repulsive sliding motion in this work.

Summarizing, we consider solutions of (1) which will exhibit either transversal intersection or attractive sliding mode on $\Sigma$. These will generally be continuous, but not differentiable, functions. Moreover, we will henceforth focus on the case in which we reach $\Sigma$ coming from $R_1$ and we will restrict to the case in which $f_1$ reaches $\Sigma$ not tangentially. To be precise, we will characterize the attractivity of $\Sigma$ from $R_1$ by the following assumption:

There exists a strictly positive constant $\delta$, such that for all $x \in R_1 \cup \Sigma$, and sufficiently close to $\Sigma$, we have

$$h^T_x(x)f_1(x) \geq \delta > 0.$$  

(11)

Observe that, since (for a trajectory in $R_1$)

$$\frac{d}{dt} h(x(t)) = h^T_x(x(t)x'(t) = h^T_x(x(t)f_1(x(t)),$$

then (11) implies that the function $h$ monotonically increases along a solution trajectory in $R_1$ (and close to $\Sigma$), until eventually the trajectory hits $\Sigma$ non-tangentially. A “discrete analog” of this property is the key to producing appropriate numerical schemes.
2. One-sided numerical methods

In some instances, the vector field \( f_1 \) (respectively, \( f_2 \)) cannot be extended smoothly outside \( R_1 \cup \Sigma \) (respectively, \( R_2 \cup \Sigma \)), or, even if it may be extended, the physical features of the model may prohibit evaluation of \( f_1 \) above (respectively, \( f_2 \) below) \( \Sigma \). The following dynamical system exemplifies this situation:

\[
x' = \begin{cases} 
  x(1-t)^{(2k+1)/2}, & k = 0, 1, 2, \ldots, \text{ when } t \leq 1 \\
  0, & \text{ when } t > 1 
\end{cases}
\]  
(12)

Note that the vector field in (12) has \( k \) continuous derivatives at \( t = 1 \); the example is an extension of one considered in [5, 6] where the authors considered the case of \( k = 0 \).

For the above reasons, we are not willing to consider numerical methods (unlike, say, [7, 8, 9]) that require that \( f_1 \) extends smoothly outside \( R_1 \cup \Sigma \), but we consider one-sided numerical procedures in which we do not need to compute the vector field \( f_1 \) outside \( R_1 \cup \Sigma \). In particular, we will study numerical procedures in which the discontinuity surface is approached from one side. These procedures compute the event or discontinuity points, and therefore they belong to the class of event driven methods (see [10]) and make sense only if on the time interval of interest there are finitely many event points.

We will study the class of general explicit Runge–Kutta (ERK) schemes and give conditions under which these methods approach the discontinuity surface from one side. As illustration of this general result, we will give sufficient conditions for a subclass of “sub-diagonal” ERK methods\(^2\) in order to satisfy these conditions. This specific class of subdiagonal ERK schemes allows for recursive arguments of proof, as well as modularity in the implementation of the schemes.

Remark 2.1. It should be said that several numerical methods have been devised for detecting event points when solving systems of ODEs. For example, the methods of [8] and [9] are in this class. Even the popular ODESUITE of Matlab codes provides “event detection” as an option. What distinguished our present construction from these other methods is that we approach the

\(^2\)In the Butcher’s tableau, only the entries in the first sub-diagonal are nonzero.
event point from one side only, and do not resort to any interpolation procedure.

We emphasize that in this work we restrict attention only to being able to locate (accurately) the point where we “hit” Σ. What to do afterwards has been treated elsewhere by a number of authors; e.g., see [11] and references there.

2.1. General Explicit Runge Kutta methods

Let us consider the general explicit Runge–Kutta (ERK) scheme defined by the Butcher’s tableau

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & \ldots & 0 \\
c_2 & a_{21} & 0 & 0 & \ldots & 0 \\
c_3 & a_{31} & a_{32} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \ldots & a_{s,s-1} & 0 \\
\hline
b_1 & b_2 & \ldots & \ldots & b_s
\end{array}
\]

Suppose we start with \( x_0 \) in the region \( R_1 \). One step of the ERK method, with time step \( \tau > 0 \), reads

\[
x_1(\tau) = x_0 + \tau \sum_{i=1}^{s} b_i f_1(y_i(\tau)),
\]

with

\[
y_1 = x_0, \quad y_i(\tau) = x_0 + \tau \sum_{j=1}^{i-1} a_{i,j} f_1(y_j(\tau)), \quad j = 2, \ldots, s.
\]

The main idea is to make sure that \( f_1 \) can be evaluated at all internal stages \( y_2, \ldots, y_s \), so that the numerical solution \( x_1(\tau) \) may be computed.

Clearly, as long as all stage values \( y_j \)'s and \( x_1 \) are in \( R_1 \), the numerical integration can proceed. Otherwise, we need to consider several different cases.

Case 1. Let us suppose (see Figure 2):

(1.a) \( h(y_i(\sigma)) \leq 0 \) for \( 0 \leq \sigma \leq \tau \) and \( i = 2, \ldots, s \);
In this case, the numerical solution with stepsize $\tau$, $x_1(\tau)$, is above $\Sigma$, while all internal stage values (for all $\sigma \in [0, \tau]$) are below $\Sigma$. This is the simplest case. Letting $x_1(\sigma) = x_0 + \sigma \sum_{i=1}^{s} b_i f_1(y_i(\sigma))$, $\sigma \in [0, \tau]$, one can use a root finding routine (say, bisection or the secant method) to compute a value $\eta$ such that the scalar function $H(\sigma) = h(x_1(\sigma))$ vanishes, that is $x_1(\eta)$ on $\Sigma$.

**Remark** In a different context, a similar approach for finding the step size such that the numerical solution lies on the discontinuity surface has been used in [12]. However, in [12] the authors use implicit RK schemes, and need to solve a nonlinear system (the stage values are parts of the unknowns); it is not clear to us that the iterative solution of the system will give approximations in the region $R_1$. On the other hand, for us, the hypothesis (1.a) guarantees that $f_1$ may be evaluated at the internal stages (14) for all $\sigma$ in $[0, \tau]$ and this implies that the iterations of the root finding routine of the scalar values function $h(x_1(\sigma))$ are well defined.

Finally, we notice that the value $\eta \in [0, \tau]$ which gives $h(x_1(\eta)) = 0$ is unique if

$$\frac{d}{d\sigma} h(x_1(\sigma)) > 0, \quad \forall \sigma \in [0, \tau],$$

and we recognize this formula as the numerical realization of (11).

**Case 2.** Let us suppose (see Figure 3):
Figure 3: Case 2: General Explicit RK method.

(2.a) \( h(y_i(\sigma)) \leq 0 \) for \( 0 \leq \sigma \leq \tau \) and \( i = 2, \ldots, s - 1 \);

(2.b) \( h(y_s(\tau)) > 0 \).

In other words, we have that the last stage value \( y_s(\tau) \) is above \( \Sigma \), while all the previous stage values are below \( \Sigma \) for all \( \sigma \in [0, \tau] \). Let us now assume that

\[
\frac{d}{d\sigma} h(y_s(\sigma)) = h_x^T(y_s(\sigma)) y_s'(\sigma) > 0, \quad \forall \sigma \in [0, \tau],
\]

which we again recognize as a monotonicity condition on the stage value function \( y_s \) (see (11)). Then, there exists a unique \( \eta \in (0, \tau) \) such that \( h(y_s(\eta)) = 0 \) and further \( h(y_s(\sigma)) < 0 \), for all \( \sigma \in [0, \eta) \). With this value of \( \eta \), we compute \( x_1(\eta) = x_0 + \eta \sum_{i=1}^{s} b_i f_1(y_i(\eta)) \) and we need to distinguish between two subcases:

(2.c) if \( h(x_1(\eta)) > 0 \), we are back to the situation treated in Case 1;

(2.d) if \( h(x_1(\eta)) \leq 0 \), then we either continue the integration with vector field \( f_1 \) (if \( h(x_1(\eta)) < 0 \)), or stop since we have found the sought point on \( \Sigma \).

The analysis of the other possibilities proceed along similar lines. 

**Case 3.** Suppose that:

(3.a) \( h(y_i(\sigma)) \leq 0 \) for \( 0 \leq \sigma \leq \tau \) and \( i = 2, \ldots, s - 2 \);
Assume that (again a monotonicity condition for the stage function $y_{s-1}$):

$$\frac{d}{d\sigma} h(y_{s-1}(\sigma)) = h^T_x (y_{s-1}(\sigma)) y'_s(\sigma) > 0, \quad \forall \sigma \in [0, \tau].$$

Then, there exists a unique $\tilde{\eta} \in (0, \tau)$ such that $h(y_{s-1}(\tilde{\eta})) = 0$. Similarly to before, we have to distinguish between two subcases:

(3.c) if $h(y_s(\tilde{\eta})) \leq 0$, then we can form $x_1(\tilde{\eta})$; if $h(x_1(\tilde{\eta})) > 0$, we will assume that

$$h(y_s(\sigma)) \leq 0 \text{ for } \sigma \in (0, \tilde{\eta})$$

in order to compute $\hat{\eta}$ such that $h(x_1(\hat{\eta}))$ vanishes;

(3.d) if $h(y_s(\tilde{\eta})) > 0$, then we go back to case 2.

All other cases, until the situation where $y_2(\tau)$ is above $\Sigma$, and $y_1(\tau)$ (that is, $x_0$) is below $\Sigma$ may be treated in much the same way. Details are clear and therefore omitted.

We stress that, as long as appropriate monotonicity assumptions hold for the stage value functions, any explicit Runge-Kutta method can approach the discontinuity surface from one side.

Next, we exemplify what properties are needed of the vector field, in order to make sure that these monotonicity properties hold. We do this, for a class of subdiagonal explicit Runge Kutta schemes of order 1 through 4.

2.2. Subdiagonal Runge Kutta Methods of Order 1 through 4

**Explicit Euler method.** Consider the explicit Euler method, ERK1, defined by the following tableau:

$$
\begin{array}{c|cc}
0 & 0 & 1 \\
\end{array}
$$

In the region $R_1$, one step of ERK1 with stepsize $\tau$ reads

$$x_1 = x_0 + \tau f_1(x_0). \quad (16)$$

If $x_1$ is in $R_1$ we continue to integrate, otherwise we are above $\Sigma$. Consider the function $x_1(\sigma) = x_0 + \sigma f_1(x_0), \ 0 \leq \sigma \leq \tau$. Trivially, this is a monotone

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3which, again, can be guaranteed by the monotonicity assumption $\frac{d}{d\sigma} h(y_s(\sigma)) > 0$
function. It is a simple observation that the function \( h(x_1(\sigma)) \) changes sign in \([0, \tau]\), and therefore there must be a value \( \eta \in [0, \tau] \) where this function has a zero. If we want to make sure that this is the only root of \( h(x_1(\sigma)) \) for \( \sigma \in [0, \tau] \), then we need that the straight line segment \( x_1(\sigma) \) intersect \( \Sigma \) just once\(^4\); this requires a control on the curvature of \( \Sigma \) with respect to the stepsize and the attractivity rate \( \delta \) of (11). This is the content of the next result.

**Theorem 2.2.** Let \( x_0 \in R_1 \) and close to \( \Sigma \). Let \( \tau > 0 \) be the stepsize of the method and let \( x_1(\sigma) = x_0 + \sigma f_1(x_0), \ 0 \leq \sigma \leq \tau \) (so that \( x_1(\tau) = x_1 \)). Let \( \tau \) be sufficiently small, and assume that there exist two strictly positive constant \( \delta \) and \( \rho \) such that

\begin{align*}
\text{(S1)} & \quad h^T_x(x_0)f_1(x_0) \geq \delta; \\
\text{(S2)} & \quad [f_1(x_0)]^T h_{xx}(x_1(\sigma))f_1(x_0) \geq -\rho, \text{ for all } \sigma \in [0, \tau]; \\
\text{(S3)} & \quad \delta - \rho \tau > 0.
\end{align*}

Then, the function \( h(x_1(\sigma)) \) is strictly increasing for \( \sigma \in [0, \tau] \). In particular:

(i) if \( h(x_1) \leq 0 \), then \( h(x_1(\sigma)) \leq 0 \) for all \( \sigma \in [0, \tau] \);

(ii) if \( h(x_1) > 0 \), then there exists a unique \( \eta \in (0, \tau) \) on \( \Sigma \).

**Proof** We observe that

\[
\frac{d}{d\sigma} h(x_1(\sigma)) = h^T_x(x_1(\sigma))x'_1(\sigma) = h^T_x(x_1(\sigma))f_1(x_0)
\]

and therefore we have

\[
\frac{d}{d\sigma} h(x_1(\sigma)) = h^T_x(x_0)f_1(x_0) + \int_0^1 [f_1(x_0)]^T h_{xx}(x_0+s(x_1(\sigma)-x_0))(x_1(\sigma)-x_0)ds,
\]

and so \( \frac{d}{d\sigma} h(x_1(\sigma)) \geq \delta - \rho \tau > 0. \)

\(^4\)This is not strictly required; since the method has 1st order accuracy, and (11) holds, any root of \( h(x_1(\sigma)) \) will do.
Explicit midpoint method. The explicit midpoint method, ERK2, is defined by the following tableau

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1 \\
\end{array}
\]

In the region \( R_1 \), one step of ERK2 with stepsize \( \tau \) reads

\[
x_1 = x_0 + \tau f_1(y_2) , \quad \text{with} \quad y_2 = x_0 + \frac{\tau}{2} f_1(x_0) ,
\]

and notice that \( y_2 \) is one step of Euler method (16) with stepsize \( \tau/2 \).

Now, suppose we have \( x_0 \in R_1 \), close to \( \Sigma \). If \( h(x_0 + \frac{\tau}{2} f_1(x_0)) \leq 0 \) and also \( h(x_1) \leq 0 \), we continue integrating this system. Otherwise, we will have to distinguish between the following two cases:

(a) \( h(y_2) \leq 0 \) but \( h(x_0 + \tau f_1(y_2)) > 0 \),

(b) \( h(y_2) > 0 \).

Case (a). Define \( y_2(\sigma) = x_0 + \frac{\sigma}{2} f_1(x_0) \), for \( \sigma \in [0, \tau] \), and assume that \( y_2(\sigma) \in R_1 \), for all \( \sigma \in [0, \tau] \); this can be guaranteed under conditions much like those of Theorem 2.2, namely: \( h_T(x_0)f_1(x_0) \geq \delta, \quad |f_1(x_0)|^T h_{xx}(y_2(\sigma))f_1(x_0) \geq -\rho \) and \( \delta - \rho \tau/2 > 0 \). In this case, take the function \( H(\sigma) = h(x_1(\sigma)) \), \( \sigma \in [0, \tau] \), where \( x_1(\sigma) = x_0 + \sigma f_1(y_2(\sigma)) \). Observe that \( H(\sigma) \) is a smooth function, taking values of opposite sign at the endpoints of the interval \([0, \tau]\).

As a consequence, there must be a (first) value, call it \( \eta \), where \( H(\eta) = 0 \). If we want that this is the unique value in \([0, \tau]\) where \( H \) vanishes\(^5\), we can give sufficient conditions to guarantee that the function \( H(\sigma) \) is monotone for \( \sigma \in [0, \tau] \). The theorem below is such a result.

**Theorem 2.3.** Consider case (a), and assume that \( h(y_2(\sigma)) \leq 0 \), for all \( \sigma \in [0, \tau] \). Further, assume that there are constants \( \delta_2 > 0 \) and \( \rho_2 > 0 \) and let \( \tau > 0 \) be sufficiently small such that the following conditions hold:

(S1) \( h_T(x_1(\sigma))f_1(y_2(\sigma)) \geq \delta_2 \), for all \( \sigma \in [0, \tau] \) ;

(S2) \( h_T(x_1(\sigma))Df_1(y_2(\sigma))f_1(x_0) \geq -\rho_2 \), for all \( \sigma \in [0, \tau] \) ;

\(^5\)Again, in this case it is not strictly necessary that there be a unique root of \( H(\sigma) \)
Then, the function $h(x_1(\sigma))$ is strictly increasing for $\sigma \in [0, \tau]$. In particular, there exists a unique $\eta \in (0, \tau)$ such that $h(x_1(\eta)) = 0$.

Proof For $\sigma \in [0, \tau]$, we have:

$$\frac{d}{d\sigma} h(x_1(\sigma)) = h^T_x(x_1(\sigma))x'_1(\sigma).$$

Since

$$x'_1(\sigma) = f_1(y_2(\sigma)) + \frac{\sigma}{2}Df_1(y_2(\sigma))f_1(x_0),$$

we get

$$h^T_x(x_1(\sigma))x'_1(\sigma) = h^T_x(x_1(\sigma))f_1(y_2(\sigma)) + \frac{\sigma}{2}h^T_x(x_1(\sigma))Df_1(y_2(\sigma))f_1(x_0)$$

and so, using (S1)-(S3) we obtain

$$\frac{d}{d\sigma} h(x_1(\sigma)) \geq \delta_2 - \frac{\sigma^2}{2}\rho_2 \geq \delta_2 - \frac{\tau^2}{2}\rho_2 > 0 \quad \square$$

Therefore, in case (a), and under the assumptions of Theorem 2.3, the scalar function $H(\sigma) = h(x_1(\sigma))$ has a unique zero which can be found by standard techniques.

Case (b). Next, consider case (b). Now, the stage value $y_2$ is already on the other side of $\Sigma$, and thus we cannot properly form $x_1$. So, we first seek a value $\eta \in (0, \tau)$ such that $y_2(\eta) \in \Sigma$. Then, if $x_1(\eta) = x_0 + \eta f_1(y_2(\eta))$ is above $\Sigma$, we are back to case (a) relatively to the stepsize $\eta$, and therefore the fact that there will exist a (unique) value $\eta \in [0, \tau]$ for which $h(x_1(\eta)) = 0$ can rest on Theorem 2.3 (the details are omitted). On the other hand, if $x_1(\eta)$ is below $\Sigma$, we continue integrating.

Heun method. This explicit Runge Kutta method of order 3, ERK3, is defined by the following tableau:

$$
\begin{array}{ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\
\frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \\
\end{array}
$$

In the region $R_1$, with stepsize $\tau$, one step of ERK3 reads

$$x_1 = x_0 + \tau \left[ \frac{1}{4} f_1(x_0) + \frac{3}{4} f_1(x_0 + \frac{2}{3} \tau f_1(x_0) + \frac{1}{3} \tau f_1(x_0)) \right], \quad (18)$$

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Figure 4: ERK3: Different cases for ERK3.

that is

\[
\begin{align*}
  x_1 &= x_0 + \tau \left[ \frac{1}{4} f_1(x_0) + \frac{3}{4} f_1(y_3) \right], \\
y_3 &= x_0 + 2\frac{\tau}{3} f_1(y_2), \\
y_2 &= x_0 + \frac{\tau}{3} f_1(x_0).
\end{align*}
\]

Notice that \( y_3 \) can be rewritten as

\[
y_3 = x_0 + 2\frac{\tau}{3} f_1(y_2) = x_0 + 2\frac{\tau}{3} f_1(x_0 + \frac{\tau}{3} f_1(x_0)),
\]

and therefore the stage value \( y_2 \) corresponds to one step of (16) with stepsize \( \tau/3 \), and \( y_3 \) corresponds to one step of (17) with stepsize \( 2\tau/3 \), instead of \( \tau \).

Now, suppose we are integrating \( x'(t) = f_1(x(t)) \), starting with \( x_0 \) below the surface \( \Sigma \), that is \( h(x_0) < 0 \), but close to it. If \( h(y_2) \leq 0 \), \( h(y_3) \leq 0 \) and \( h(x_1) \leq 0 \), we continue integrating this system. Otherwise, we will have to distinguish between the three cases (see Figure 4):

(a) \( h(y_2) \leq 0, h(y_3) \leq 0 \) but \( h(x_1) > 0 \),

(b) \( h(y_2) \leq 0 \) but \( h(y_3) > 0 \),

(c) \( h(y_2) > 0 \).

Case (a). Define \( y_2(\sigma) = x_0 + \frac{\sigma}{3} f_1(x_0) \), for \( \sigma \in [0, \tau] \), and assume that \( y_2(\sigma) \in R_1 \), for all \( \sigma \in [0, \tau] \), that is \( h(y_2(\sigma)) \leq 0 \). This can be guaranteed by conditions like those in Theorem 2.2, with the only change there being \( \delta - \rho \tau / 3 > 0 \). In this case, the function \( y_3(\sigma) = x_0 + \frac{\sigma}{3} f_1(y_2(\sigma)) \), for all \( \sigma \in [0, \tau] \), is well defined. Further, assume that \( h(y_3(\sigma)) \leq 0, \forall \sigma \in [0, \tau] \); this
can be guaranteed by conditions similar to those in Theorem 2.3, namely: for all \( \sigma \in [0, \tau] \), we have \( h_T^x(y_3(\sigma)) f_1(y_2(\sigma)) \geq \delta_2, h_T^x(y_3(\sigma)) D f_1(y_2(\sigma)) f_1(x_0) \geq -\rho_2 \), and \( \delta_2 - \frac{\tau}{2} \rho_2 > 0 \). In this case, for all \( \sigma \in [0, \tau] \), the function \( H(\sigma) = h(x_1(\sigma)) \) is well defined, where

\[
x_1(\sigma) = x_0 + \sigma \left( \frac{1}{4} f_1(x_0) + \frac{3}{4} f_1(y_3(\sigma)) \right).
\]

Since \( H(\cdot) \) is a smooth function and takes values of opposite sign at the endpoints of the interval \([0, \tau]\), there must be a (first) value, call it \( \eta \), where \( H(\eta) = 0 \). We now give sufficient conditions guaranteeing that this is the unique value in \([0, \tau]\) where \( H \) vanishes\(^6\).

**Theorem 2.4.** Consider case (a), and assume that \( h(y_2(\sigma)) \leq 0 \) and \( h(y_3(\sigma)) \leq 0 \), for all \( \sigma \in [0, \tau] \). Further, let there be constants \( \delta_3 > 0 \), \( \rho_3 \geq 0 \) and \( \eta_3 > 0 \), and let \( \tau > 0 \) be sufficiently small so that the following conditions hold:

(S1) \( h_T^x(x_1(\sigma))[f_1(x_0) + 3 f_1(y_3(\sigma))] / 4 \geq \delta_3 \), for all \( \sigma \in [0, \tau] \);

(S2) \( h_T^x(x_1(\sigma)) D f_1(y_3(\sigma)) f_1(y_2(\sigma)) \geq -\rho_3 \), for all \( \sigma \in [0, \tau] \);

(S3) \( h_T^x(x_1(\sigma)) D f_1(y_3(\sigma)) D f_1(y_2(\sigma)) f_1(x_0) \geq -\eta_3 \), for all \( \sigma \in [0, \tau] \);

(S4) \( \delta_3 - \frac{\tau}{2} \rho_3 - \frac{\tau^2}{6} \eta_3 > 0 \).

Then, the function \( h(x_1(\sigma)) \) is strictly increasing for \( \sigma \in [0, \tau] \). In particular, there exists a unique \( \eta \in (0, \tau) \) such that \( h(x_1(\eta)) = 0 \).

**Proof** The proof amounts to a straightforward differentiation. For \( \sigma \in [0, \tau] \), we have:

\[
\frac{d}{d\sigma} h(x_1(\sigma)) = h_T^x(x_1(\sigma)) x'_1(\sigma),
\]

\[
x'_1(\sigma) = \frac{1}{4} (f_1(x_0) + 3 f_1(y_3(\sigma))) + \frac{3\sigma}{4} D f_1(y_3(\sigma)) y_3'(\sigma),
\]

and

\[
y_3'(\sigma) = \frac{2}{3} f_1(y_2(\sigma)) + \frac{2\sigma}{9} D f_1(y_2(\sigma)) f_1(x_0).
\]

\(^6\)Again, in this case it is not strictly necessary that there be a unique root of this \( H(\sigma) \)
So, using (S1)-(S4), we obtain

\[
\frac{d}{d\sigma} h(x_1(\sigma)) \geq \delta_3 - \frac{\sigma}{2} \rho_3 - \frac{\sigma^2}{6} \eta_3 > 0
\]

\[\square\]

**Case (b).** Now \( h(y_3) > 0 \). Assume that \( y_2(\sigma) \in R_1 \) (that is, \( h(y_2(\sigma)) \leq 0 \)), for all \( \sigma \in [0, \tau] \); this can be guaranteed by conditions like those in Theorem 2.2. At this point, we look for the unique value \( \eta \in (0, \tau) \) such that \( h(y_3(\eta)) = 0 \), and form \( x_1(\eta) = x_0 + \eta \left[ \frac{1}{6} f_1(x_0) + \frac{1}{3} f_1(y_2) + \frac{1}{3} f_1(y_3) + \frac{1}{6} f_1(y_4) \right] \). The existence and uniqueness of this value \( \eta \) can be guaranteed by a result just like Theorem 2.3. If \( h(x_1(\eta)) > 0 \), we are back to consider a situation like in case (a), relatively to the interval \([0, \eta]\), otherwise we continue to integrate.

**Case (c).** In this case, \( y_2 \) is on the other side of \( \Sigma \): \( h(y_2) > 0 \). In this situation we can use Theorem 2.2 to guarantee that \( h(y_2(\sigma)) \) is an increasing function for \( \sigma \in [0, \tau] \), and so there exists a unique \( \eta \in [0, \tau] \) such that \( h(y_3(\eta)) = 0 \) where \( y_2(\eta) = x_0 + \frac{2}{3} \eta f_1(y_1(\eta)) \). If \( h(y_3(\eta)) > 0 \), relatively to the stepsize \( \eta \) (instead of \( \tau \)), we are back to case (b); otherwise, if \( h(y_3(\eta)) \leq 0 \), we are back to case (a).

**Classical Runge Kutta Method.** The last one-sided integration scheme we propose is based on the classical explicit Runge Kutta method of order 4, ERK4, defined by the following tableau:

\[
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
\]

In \( R_1 \), one step of length \( \tau \) of ERK4 is

\[
x_1(\tau) = x_0 + \tau \left[ \frac{1}{6} f_1(x_0) + \frac{1}{3} f_1(y_2) + \frac{1}{3} f_1(y_3) + \frac{1}{6} f_1(y_4) \right] \tag{19}
\]

where

\[
y_2 = x_0 + \frac{\tau}{2} f_1(x_0), \\
y_3 = x_0 + \frac{\tau}{2} f_1(y_2) = x_0 + \frac{\tau}{2} f_1(x_0 + \frac{\tau}{2} f_1(x_0)), \\
y_4 = x_0 + \tau f_1(y_3) = x_0 + \tau f_1(x_0 + \frac{\tau}{2} f_1(x_0 + \frac{\tau}{2} f_1(x_0))).
\]
Observe that the stage value $y_2$ corresponds to (16), and $y_3$ to (17), with stepsizes $\frac{\tau}{2}$ instead of $\tau$.

Now, suppose we are integrating $x'(t) = f_1(x(t))$, starting with $x_0 \in R_1$, but close to $\Sigma$. If $h(y_2) \leq 0$, $h(y_3) \leq 0$, $h(y_4) \leq 0$ and also $h(x_1) \leq 0$, we continue integrating this system. Otherwise, we will have to distinguish between the four cases:

(a) $h(y_2) \leq 0$, $h(y_3) \leq 0$, $h(y_4) \leq 0$ but $h(x_1) > 0$,

(b) $h(y_2) \leq 0$, $h(y_3) \leq 0$, but $h(y_4) > 0$,

(c) $h(y_2) \leq 0$ but $h(y_3) > 0$,

(d) $h(y_2) > 0$.

Figure 5: Different cases for ERK4
Case (a). Assume that for all \( \sigma \in [0, \tau] \), we have \( h(y_2(\sigma)) \leq 0 \), \( h(y_3(\sigma)) \leq 0 \), and \( h(y_4(\sigma)) \leq 0 \), where

\[
\begin{align*}
y_2(\sigma) &= x_0 + \frac{\eta}{2} f_1(x_0), \\
y_3(\sigma) &= x_0 + \frac{\eta}{2} f_1(y_2(\sigma)), \\
y_4(\sigma) &= x_0 + \sigma f_1(y_3(\sigma)).
\end{align*}
\]

Observe that sufficient conditions to guarantee that \( h(y_2(\sigma)) \leq 0 \) and \( h(y_3(\sigma)) \leq 0 \) can be obtained much like in Theorems 2.2 and 2.3. To guarantee that \( h(y_4(\sigma)) \leq 0 \), we need a result similar to Theorem 2.4, which we state without proof.

**Theorem 2.5.** Consider case (a), and assume that \( h(y_2(\sigma)) \leq 0 \) and \( h(y_3(\sigma)) \leq 0 \), for all \( \sigma \in [0, \tau] \). Let there be constants \( \delta_4 > 0 \), \( \rho_4 > 0 \) and \( \eta_4 > 0 \), and let \( \tau > 0 \) be sufficiently small so that the following conditions hold:

\[
\begin{align*}
(S1) & \quad h^T_x(y_4(\sigma)) f_1(y_3(\sigma)) \geq \delta_4, \quad \text{for all } \sigma \in [0, \tau]; \\
(S2) & \quad h^T_x(y_4(\sigma)) D f_1(y_3(\sigma)) f_1(y_2(\sigma)) \geq -\rho_4, \quad \text{for all } \sigma \in [0, \tau]; \\
(S3) & \quad h^T_x(y_4(\sigma)) D f_1(y_3(\sigma)) D f_1(y_3(\sigma)) f_1(x_0) \geq -\eta_4, \quad \text{for all } \sigma \in [0, \tau]; \\
(S4) & \quad \delta_4 - \frac{\tau}{2} \rho_4 - \frac{\tau^2}{8} \eta_4 > 0.
\end{align*}
\]

Then, the function \( h(y_4(\sigma)) \) is strictly increasing for \( \sigma \in [0, \tau] \).

Under the conditions of Theorem 2.5, \( y_2(\sigma), y_3(\sigma) \) and \( y_4(\sigma) \) are below \( \Sigma \) for all \( \sigma \in [0, \tau] \) and so the function \( H(\sigma) = h(x_1(\sigma)) \), \( \sigma \in [0, \tau] \) is well defined, where

\[
x_1(\sigma) = x_0 + \sigma \left[ \frac{1}{6} f_1(x_0) + \frac{1}{3} f_1(y_2(\sigma)) + \frac{1}{3} f_1(y_3(\sigma)) + \frac{1}{6} f_1(y_4(\sigma)) \right].
\]

Since \( H(\cdot) \) is smooth and takes values of opposite signs at 0 and \( \tau \), there is a (first) value, call it \( \eta \) such that \( h(x_1(\eta)) = 0 \). Sufficient conditions guaranteeing that this is the only root of \( H(\cdot) \) in \([0, \tau]\) can of course be given, but it is not strictly necessary that \( H(\cdot) \) vanishes at only one point, any zero will do.

Case (b). Assuming that, for all \( \sigma \in [0, \tau] \), \( h(y_2(\sigma)) \leq 0 \) and \( h(y_3(\sigma)) \leq 0 \), which we can guarantee much like in Theorems 2.2 and 2.3, then the function \( y_4(\sigma) \) is well defined for all \( \sigma \in [0, \tau] \). Thus, we will look for the unique value
η for which \( h(y_4(\eta)) = 0 \). Existence and uniqueness of η can be guaranteed by Theorem 2.5. At this point, we consider the scheme (19) with η replacing τ there. If \( h(x_1(\eta)) > 0 \), we are back to case (a), relatively to the interval \([0, \eta]\) and can thus find a value \( \bar{\eta} \in [0, \eta] \) such that \( h(x_1(\bar{\eta})) = 0 \); otherwise if \( h(x_1(\eta)) \leq 0 \), we continue integrating.

Case (c). Now we need to assume that, for all \( \sigma \in [0, \tau] \), \( h(y_2(\sigma)) \leq 0 \), which we can guarantee like in Theorem 2.2. Then, the function \( y_3(\sigma) \) is well defined for all \( \sigma \in [0, \tau] \) and we look for the unique value \( \tau_1 < \tau \) for which \( h(y_3(\tau_1)) = 0 \), which we can guarantee as in Theorem 2.3. Now, on \([0, \tau_1]\), also the function \( y_4(\sigma) \) is well defined. If \( h(y_4(\tau_1)) > 0 \), we are back to case (b), relatively to the stepsize \( \tau_1 \) instead of \( \tau \); otherwise we are back to case (a).

Case (d). Here, we first have to take the step \( \tau_1 < \tau \) for which \( h(y_2(\tau_1)) = 0 \), which we can do as in Theorem 2.2. So, \( y_3(\sigma) \) is well defined for all \( \sigma \in [0, \tau_1] \) and \( h(y_3(\tau_1)) > 0 \). Thus we are back to Case (c), relatively to the stepsize \( \tau_1 \) instead of \( \tau \).

Remark 2.6. When implementing the method (19), in fact any of the general ERK methods, we need to check the position of the stage values and the final approximation with respect to the discontinuity surface \( \Sigma \). Of course, we must always start with the first stage value and proceed (by possibly progressively decreasing the stepsize) to the next stage values and then the final approximation.

Remark 2.7. Finally, we observe that in case the discontinuous differential system is stiff, then one sided ERK methods will require a small step size, for stability reasons. On the other hand, an implicit method will require solution of a nonlinear (algebraic) system, and thus \( f_1 \) will need to be evaluated at the iterates arising during solution of such system; in general, these may be values above \( \Sigma \).

3. Numerical Tests

In this section we will apply our scheme to two different examples. The first example is one where \( f_1 \) extends smoothly from \( R_1 \) into \( R_2 \), and use this example to check accuracy of the computed event point. Our second example, instead, is effectively (12) rewritten in autonomous form; in this case, the vector field \( f \) has only limited smoothness and we will see that this leads to unavoidable loss of order.
Example 1.

Consider the discontinuous system from [13], with vector fields

\[ f_1(x) = \left(-x_1 + \frac{x_2}{1.2-x_2}\right), \quad f_2(x) = \left(-x_1 - \frac{x_2}{0.8+x_2}\right). \]

The discontinuity surface \( \Sigma \) is the line \( h(x_1, x_2) = x_2 - 0.2 = 0 \). A plot of the solution on the time interval \([0, 10]\) and initial conditions (ICs) \((x_1(0), x_2(0)) = (-0.4, -0.5)\) is shown in the Figure 6. Another plot of the solution up to \( \Sigma \), for ICs \((x_1(0), x_2(0)) = (0.4, -0.5)\), is in Figure 7.

For the two different ICs, we computed (at very high accuracy) the intersection points and they are

\[
\begin{align*}
\text{ICs } (-0.4, -0.5) & \rightarrow x_1 = -0.49889048306999, \quad x_2 = 0.2, \\
\text{ICs } (0.4, -0.5) & \rightarrow x_1 = 0.10921295035161, \quad x_2 = 0.2.
\end{align*}
\]

We have used the four one-sided subdiagonal ERK methods considered previously, with step size \( \tau = 0.01 \), to compute the point where the numerical trajectory hits \( \Sigma \). In Tables 1 and 2, we report on the numerical solution at this event point, for the two different sets of ICs, for these four methods.

The column “Final Time” denotes the time value at which the corresponding numerical solution hits the discontinuity surface.
Table 1: ICs $(-0.4, -0.5)$

<table>
<thead>
<tr>
<th>Method</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>Error</th>
<th>Ratio Error</th>
<th>Final Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>-0.50268467261891</td>
<td>0.2</td>
<td>3.7942E-3</td>
<td>0.5815</td>
<td></td>
</tr>
<tr>
<td>ERK2</td>
<td>-0.4988786523447</td>
<td>0.2</td>
<td>7.3823E-6</td>
<td>5.14E+2</td>
<td>0.5805</td>
</tr>
<tr>
<td>ERK3</td>
<td>-0.49889047489584</td>
<td>0.2</td>
<td>8.1742E-9</td>
<td>9.03E+2</td>
<td>0.5803</td>
</tr>
<tr>
<td>ERK4</td>
<td>-0.49889048306839</td>
<td>0.2</td>
<td>1.5999E-12</td>
<td>5.01E+3</td>
<td>0.5803</td>
</tr>
</tbody>
</table>

Table 2: ICs $(0.4, -0.5)$

<table>
<thead>
<tr>
<th>Method</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>Error</th>
<th>Ratio Error</th>
<th>Final Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.10440596383147</td>
<td>0.2</td>
<td>4.807E-3</td>
<td>1.3095</td>
<td></td>
</tr>
<tr>
<td>ERK2</td>
<td>0.10920429334411</td>
<td>0.2</td>
<td>8.657E-6</td>
<td>5.55E+2</td>
<td>1.3074</td>
</tr>
<tr>
<td>ERK3</td>
<td>0.10921297100861</td>
<td>0.2</td>
<td>2.066E-8</td>
<td>4.19E+2</td>
<td>1.3074</td>
</tr>
<tr>
<td>ERK4</td>
<td>0.10921295040386</td>
<td>0.2</td>
<td>5.225E-11</td>
<td>3.95E+2</td>
<td>1.3074</td>
</tr>
</tbody>
</table>

Figure 7: Solution with ICs $(0.4, -0.5)$. 
The column denoted by “Ratio Error” gives the ratio between the global error of the method and the one of the method of previous smaller order. From these, we can see that the methods behave consistently with their expected order, in that there is a gain of $O(\tau)$ accuracy as we increase the order.

3.2. Example 2.

Now we consider the discontinuous system in (12), rewritten in the form of (1), with $f$ being a $C^k$ function. More precisely, we have

$$f_1(x_1, x_2) = \left( \frac{x_1(1 - x_2)^{2k+1}}{1}, \quad f_2(x_1, x_2) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right),$$

and consider initial conditions $x_1(0) = 0.5$ and $x_2(0) = 0$. The discontinuity surface $\Sigma$ is now the line $h(x_1, x_2) = x_2 - 1 = 0$, separating the two regions $R_1 = \{x : x_2 < 1\}$ and $R_2 = \{x : x_2 > 1\}$. The exact solution of the differential system at $x_2 = 1$ is given by $x_1(1) = x_1(0) \exp(\frac{2}{2k+3})$. Table 3 shows the numerical solution, for $k = 0$, by means of ERK1 and ERK2 methods at the discontinuity point. Here we show the global errors for these two methods ($GE_{ERK1}$ and $GE_{ERK2}$, respectively) obtained by using halved stepsize. We started with $\tau = 0.014$, in order to avoid to arrive on $t = 1$ by an integer multiple of $\tau$, so that a root finding routine is required. In the columns $RF_{ERK1}$ and $RF_{ERK2}$ we show the reduction factors of the two methods (that is the ratio between the global error obtained by using $\tau$ and the one obtained by using $\tau/2$). Such a behaviour seems to indicate that ERK1 has order 1, while ERK2 does not have order 2. A reduction of the error similar to the one observed for ERK2 occurs for ERK3 and ERK4. [It is a simple explicit verification to show that the error is behaving as it should, given the form of the exact solution of this problem.]

Next, consider the case $k = 3$ in (20). In Table 4 we show the global errors of the four subdiagonal RK methods considered in this work, at the discontinuity point $t = 1$. Quite clearly, all methods display the expected order of accuracy.

References

Table 3: ERK1 and ERK2 Methods: $k = 0$ in (20)

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$GE_{ERK1}$</th>
<th>$RF_{ERK1}$</th>
<th>$GE_{ERK2}$</th>
<th>$RF_{ERK2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.014</td>
<td>2.9235E-3</td>
<td>7.2817E-5</td>
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<td></td>
</tr>
<tr>
<td>0.014/2</td>
<td>1.5635E-3</td>
<td>3.2566E-5</td>
<td>2.23</td>
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</tr>
<tr>
<td>0.014/4</td>
<td>7.9718E-4</td>
<td>1.0172E-5</td>
<td>3.20</td>
<td></td>
</tr>
<tr>
<td>0.014/8</td>
<td>4.0491E-4</td>
<td>1.0172E-5</td>
<td>3.20</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: ERK1, ERK2, ERK3, ERK4 Methods: $k = 3$ in (20)

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$GE_{ERK1}$</th>
<th>$RF_{ERK1}$</th>
<th>$GE_{ERK2}$</th>
<th>$RF_{ERK2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.014</td>
<td>3.800E-3</td>
<td>4.299E-6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.014/2</td>
<td>1.900E-3</td>
<td>1.078E-6</td>
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<tr>
<td>0.014/4</td>
<td>9.574E-4</td>
<td>2.698E-7</td>
<td>3.99</td>
<td></td>
</tr>
<tr>
<td>0.014/8</td>
<td>4.784E-4</td>
<td>6.750E-8</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$GE_{ERK3}$</th>
<th>$RF_{ERK3}$</th>
<th>$GE_{ERK4}$</th>
<th>$RF_{ERK4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.014</td>
<td>5.488E-09</td>
<td>8.207E-11</td>
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<tr>
<td>0.014/2</td>
<td>7.017E-10</td>
<td>5.585E-12</td>
<td>14.69</td>
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</tr>
<tr>
<td>0.014/4</td>
<td>8.877E-11</td>
<td>3.598e-13</td>
<td>15.52</td>
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<tr>
<td>0.014/8</td>
<td>1.116E-11</td>
<td>2.409E-14</td>
<td>14.93</td>
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</tbody>
</table>


