SMOOTHNESS AND PERIODICITY OF SOME MATRIX DECOMPOSITIONS*

JANN-LONG CHERN† AND LUCA DIECI‡

Abstract. In this work we consider smooth orthonormal factorizations of smooth matrix-valued functions of constant rank. In particular, we look at Schur, singular value, and related decompositions. Furthermore, we consider the case in which the functions are periodic and study periodicity of the factors. We allow for eigenvalues and singular values to coalesce.

Key words. constant rank, orthonormal factorizations, periodic matrices

AMS subject classifications. 15A, 65F, 65L

1. Introduction. In the recent paper [6], Dieci and Eirola considered smooth orthonormal factorizations of smooth time-dependent matrix-valued functions. The purpose of the present work is to further the study of [6] in two distinct directions:

(i) extend some of the results of [6] to the case in which the function to be factored has constant rank;

(ii) consider the case in which the function is periodic and study periodicity of the factors.

Thus, we consider a $k$ times differentiable matrix-valued function of real variable $t \rightarrow A(t)$ and write $A \in C^k(\mathbb{R}, \mathbb{F}^{m \times n})$, $k \geq 0$, where we have $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. We can think of $t$ as time, and in practice $t$ may belong to an interval (open or close), or the half line, rather than the whole real line, but this has no bearing on our results. If all entries of $A$ are periodic of period $\tau$, then we will write $A \in C^k_{\tau}(\mathbb{R}, \mathbb{F}^{m \times n})$. We call $Q \in \mathbb{F}^{m \times n}$ orthonormal if $Q^*Q = I$; in case $m = n$, we call $Q$ unitary if $\mathbb{F} = \mathbb{C}$ and orthogonal if $\mathbb{F} = \mathbb{R}$. Also, in what follows, by $\Lambda(B)$ we will indicate the set of eigenvalues of the matrix $B$.

The study of functions with constant rank is important in applications related to differential algebraic systems; for example, see [16, 5, 15] and especially [2, sections 2.4–2.5]. Periodic matrix-valued functions arise quite often in the study of dynamical systems (e.g., see [20] and [11]), and it is clearly of interest being able to understand not just the smoothness of the factors relative to their factorizations, but also the periodicity of these factors.

Early study of both issues appear in the work of Sibuya; see [19]. Sibuya’s study is about block diagonalization of matrix-valued functions with (two) disjoint groups of eigenvalues, and he studied both smoothness and periodicity of the diagonalizing transformation. Some of Sibuya’s results were later somewhat improved by work of Eremenko (see [7]), but—as far as we could determine—Sibuya’s periodicity results basically are still the best available. More recent study of factorization of analytic functions with constant rank is implicit in the work of Bunse-Gerstner et al. (see [3]),

---

*Received by the editors March 22, 1999; accepted for publication (in revised form) by A. Bunse-Gerstner June 6, 2000; published electronically October 31, 2000. This work was supported in part under NSF grants DMS-9625813 and DMS-9973266.

†Department of Mathematics, National Central University, Chung-Li 32054, Taiwan, Republic of China (chern@math.ncu.edu.tw).

‡School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 (dieci@math.gatech.edu).
who considered analytic singular value decompositions (SVD) for analytic matrix functions. Some effort in the smooth, nonanalytic case for smooth SVDs of constant rank functions can be found in the Ph.D. thesis of Pütz [17], who also tackles computational issues.

Our chief contributions are twofold. On the one hand, we improve upon existing results, and give new results, about smoothness of constant rank functions, in particular for the SVD and related factorizations such as Takagi’s factorization and generalized SVD. On the other hand, we give new results in the periodic case in the case of coalescing eigenvalues (singular values): we classify periodicity of the eigendecomposition in the Hermitian case and similarly for the SVD.

An outline of the paper is as follows. In the next section, we consider smooth factorizations for smooth matrix-valued functions of constant rank. Then, in section 3 we consider the periodic case. To maintain focus on these two separate issues, we found it convenient to split these two topics.

Remark 1.1. In [6], under some nondegeneracy assumptions, differential equations were derived for the factors of the various decompositions examined. We have also derived differential equations for (some of) the decompositions of section 2 of the present work. However, we opted for purely algebraic proofs of the results of section 2, since the arguments used are more in tune with those adopted to prove the periodicity results of section 3 (and see also [19]), and it does not seem easy to use the differential equations to obtain the periodicity results.

2. Orthonormal decompositions in the constant rank case. The most important decomposition of this section is the SVD for functions of constant rank, Theorem 2.4, from which most other decompositions follow. In order to prove this result, we will make use of the following two lemmas, both of which are easy to prove, and the first is certainly well known (e.g., see [6]).

Lemma 2.1 (QR decomposition). Let \( A \in C^k(\mathbb{R}, \mathbb{F}^{m \times n}) \), \( m \geq n \), \( k \geq 0 \), and \( A(t) \) be full rank for all \( t \). Then \( A \) admits a factorization \( A(t) = Q(t)R(t) \) for all \( t \), where \( Q \in C^k(\mathbb{R}, \mathbb{F}^{m \times n}) \) is orthonormal and \( R(t) \in C^k(\mathbb{R}, \mathbb{F}^{n \times n}) \) is upper triangular. The factorization can be made unique by requiring \( R \) to have positive diagonal entries (real even in the case \( \mathbb{F} = \mathbb{C} \)).

Proof. This follows at once upon using the standard Gram–Schmidt’s process on the columns of \( A \).

Lemma 2.2 (invariant subspaces lemma). Let \( A \in C^k(\mathbb{R}, \mathbb{F}^{n \times n}) \) be Hermitian (symmetric if \( \mathbb{F} = \mathbb{R} \)). Let \( Q_i \in C^k(\mathbb{R}, \mathbb{F}^{n \times n}) \), \( i = 1, \ldots, p \), be orthonormal representations of separated invariant subspaces of \( A \); that is, for fixed \( p \) and dimensions \( n_i \), each \( Q_i \) is orthonormal, \( n_i + \cdots + n_p \leq n \), and

\[
AQ_i = Q_iX_i, \quad \Lambda(X_i) \cap \Lambda(X_j) = \emptyset, \quad i \neq j, \quad \text{for all } t.
\]

Then the matrix valued function \( Q := [Q_1, \ldots, Q_p] \) is orthonormal.

Proof. We need to verify that \( Q_j^*Q_i = 0 \) for \( i \neq j \). Using (2.1) and \( X_i^* = X_i \), we have

\[
X_j(Q_j^*Q_i) = Q_j^*A^*Q_i = Q_j^*AQ_i = (Q_j^*Q_i)X_i.
\]

From this we have the Lyapunov equation for \( Q_j^*Q_i \),

\[
(Q_j^*Q_i)X_i - X_j(Q_j^*Q_i) = 0,
\]

which has the unique solution \( Q_j^*Q_i = 0 \) since \( \Lambda(X_i) \cap \Lambda(X_j) = \emptyset \) for all \( t \) (see [10]).
The next result was proven in [6] when $k \geq 1$; our technique here is different and is based on [19].

**Theorem 2.3.** Let $A \in C^k(\mathbb{R}, F^{n \times n})$, $k \geq 0$, and assume that (with $p$ fixed) $A(t) = \Lambda(t) \cup \cdots \cup \Lambda_p$, where $\Lambda(t) \cap \Lambda_p = \emptyset$ for all $t$ and $i \neq j$, $i, j = 1, \ldots, p$. Further, in case $F = \mathbb{R}$, we will assume that $\det(\Lambda_i) \in \mathbb{R}$ for all $i$. (This ensures that complex conjugate eigenvalues are grouped together.) Then there exists unitary (orthogonal if $F = \mathbb{R}$) $Q \in C^k(\mathbb{R}, F^{n \times n})$ such that

$$Q^*(t)A(t)Q(t) = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1p} \\ 0 & T_{22} & \cdots & T_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{pp} \end{bmatrix},$$

where $\Lambda(T_{ii}) = \Lambda_i$, $i = 1, \ldots, p$. Moreover, if $A$ is normal,\(^1\) then $T_{ij} = 0$, $i \neq j$, and $T_{ii}$, $i = 1, \ldots, p$, are also normal.

**Proof.** Under the stated assumptions, [19, Theorem 3 and Remark 3] give the block diagonalization

$$S^{-1}(t)A(t)S(t) = \text{diag}(E_{11}, \ldots, E_{pp}),$$

where $S \in C^k(\mathbb{R}, F^{n \times n})$ and the diagonal blocks $E_{ii}$ correspond to the eigenvalues $\Lambda_i$, for $i = 1, \ldots, p$. By Lemma 2.1, we can choose $C^k$ unitary $Q$ and upper triangular $R$ such that $S = QR$. It is now enough to block partition $R$ according to the partitioning of (2.3) to obtain (2.2). Now consider the case of normal $A$. Let $T := Q^*AQ$; obviously $T$ is normal. Now, partition $T$ in (2.2) as $T = \begin{bmatrix} T_{11} & C \\ 0 & T \end{bmatrix}$ and use $TT^* = T^*T$ to obtain

$$C^*T_{11} - T^*C = 0 \quad \text{for all} \quad t,$$

whose only solution is $C = 0$ (since $\Lambda(T_{11}) \cap \Lambda(\hat{T}) = \emptyset$ for all $t$). An obvious induction argument completes the proof. \(\square\)

We are now ready for the SVD of a constant rank matrix-valued function.

**Theorem 2.4 (SVD).** Let $A \in C^k(\mathbb{R}, F^{m \times n})$, $m \geq n$ and $k \geq 0$, have constant rank: $\text{rank}(A(t)) = n - r$ for all $t$, $r$ fixed: $0 \leq r \leq n - 1$. Then there exist unitary (orthogonal if $F = \mathbb{R}$) $U \in C^k(\mathbb{R}, F^{m \times m})$ and $V \in C^k(\mathbb{R}, F^{n \times n})$ such that

$$U^*(t)A(t)V(t) = S = \begin{bmatrix} S_+ & 0 \\ 0 & 0 \end{bmatrix},$$

where $S_+ \in C^k(\mathbb{R}, F^{(n-r) \times (n-r)})$ is Hermitian (symmetric if $F = \mathbb{R}$) positive definite.

Further, suppose that the continuous eigenvalues of $S_+$ (i.e., singular values of $A$), $\lambda_1, \ldots, \lambda_{n-r}$, satisfy

$$\liminf_{\tau \to 0} \frac{\lambda_i(t + \tau) - \lambda_j(t + \tau)}{|\tau^e|} \in (0, \infty)$$

for some $e \leq k$ and for all $t$ and $i \neq j$. Then there exists unitary (orthogonal if $F = \mathbb{R}$) $Q \in C^k(\mathbb{R}, F^{(n-r) \times (n-r)})$ such that $Q^*S_+Q = \text{diag}(\lambda_1, \ldots, \lambda_{n-r})$. The singular values can be taken to be $C^k$ functions.

**Proof.** Consider the following Hermitian function:

$$B(t) = \begin{bmatrix} 0 & A(t) \\ A^*(t) & 0 \end{bmatrix}.$$
Then \( B \in C^k(\mathbb{R}, \mathbb{F}^{(m+n) \times (m+n)}) \), and it is easily verified that

\[
(2.7) \quad \text{if } \lambda(t) \in \Lambda(B(t)) \Rightarrow -\lambda(t) \in \Lambda(B(t)) \quad \text{for all } t.
\]

We also have from (2.6)

\[
(2.8) \quad \text{rank } B(t) = 2 \text{rank } A(t) = 2(n - r) \quad \text{for all } t.
\]

Moreover, from Theorem 2.3, there exists unitary \( Q \in C^k(\mathbb{R}, \mathbb{F}^{(m+n) \times (m+n)}) \) such that

\[
(2.9) \quad Q^*(t)B(t)Q(t) = \begin{bmatrix}
S_+(t) & 0 & 0 \\
0 & S_-(t) & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

where \( S_+, S_- \in C^k(\mathbb{R}, \mathbb{F}^{(n-r) \times (n-r)}) \) are Hermitian, and \( S_+/S_- \) comprise all the positive/negative eigenvalues, respectively \((S_+/S_- \text{ are positive/negative definite})\). Column partition \( Q \) according to (2.9), \( Q(t) = [Q_1(t) \quad Q_2(t) \quad Q_3(t)] \), and let

\[
W_1(t) = Q_1(t) = \begin{bmatrix}
X(t) \\
Y(t)
\end{bmatrix}, \quad W_2(t) = \begin{bmatrix}
X(t) \\
-Y(t)
\end{bmatrix},
\]

where \( X \in C^k(\mathbb{R}, \mathbb{F}^{m \times (n-r)}) \), \( Y \in C^k(\mathbb{R}, \mathbb{F}^{n \times (n-r)}) \), and \( W_1, W_2 \in C^k(\mathbb{R}, \mathbb{F}^{(m+n) \times (n-r)}) \) are orthonormal. Let

\[
(2.10) \quad W(t) = [W_1(t) \quad W_2(t) \quad Q_3(t)],
\]

so that

\[
BW_1 = W_1S_+, \quad BW_2 = W_2(-S_+), \quad BQ_3 = Q_30.
\]

Upon using Lemma 2.2, we see that \( W(t) \) is unitary for all \( t \) and

\[
(2.11) \quad W^*(t)B(t)W(t) = \begin{bmatrix}
S_+(t) & 0 & 0 \\
0 & -S_+(t) & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Now, we set

\[
(2.12) \quad U_1(t) = \sqrt{2}X(t), \quad V_1(t) = \sqrt{2}Y(t),
\]

so that \( U_1 \in C^k(\mathbb{R}, \mathbb{F}^{m \times (n-r)}) \) and \( V_1 \in C^k(\mathbb{R}, \mathbb{F}^{n \times (n-r)}) \) are orthonormal and

\[
(2.13) \quad A(t)V_1(t) = U_1(t)S_+(t) \text{ and } A^*(t)U_1(t) = V_1(t)S_+(t) \quad \text{for all } t.
\]

To complete the proof of (2.4), we need to get smooth orthonormal representations for the kernel of the row and column space of \( A \). We proceed as follows. Since \( A^*A \in C^k(\mathbb{R}, \mathbb{F}^{m \times m}) \), \( AA^* \in C^k(\mathbb{R}, \mathbb{F}^{m \times m}) \) are both of rank \( n-r \), by Theorem 2.3 there exist unitary (orthogonal if \( \mathbb{F} = \mathbb{R} \)) \( Q_1 \in C^k(\mathbb{R}, \mathbb{F}^{n \times n}) \) and \( Q_2 \in C^k(\mathbb{R}, \mathbb{F}^{m \times m}) \) such that

\[
Q_1^*(t)A^*(t)A(t)Q_1(t) = \begin{bmatrix}
M_1(t) & 0 \\
0 & 0
\end{bmatrix}, \quad Q_2^*(t)A(t)A^*(t)Q_2(t) = \begin{bmatrix}
M_2(t) & 0 \\
0 & 0
\end{bmatrix},
\]
with $M_1, M_2 \in C^k(\mathbb{R}, \mathbb{F}^{(n-r) \times (n-r)})$, Hermitian and nonsingular. Partition

$$Q_1(t) = \begin{bmatrix} Q_{11}(t) & V_2(t) \end{bmatrix}, \quad Q_2(t) = \begin{bmatrix} Q_{22}(t) & U_2(t) \end{bmatrix},$$

so that we have

$$(U_2^*(t)A(t)) (U_2^*(t)A(t))^* = 0, \quad (A(t)V_2(t))^* (A(t)V_2(t)) = 0,$$

and thus

$$U_2^*(t)A(t) = 0, \quad A(t)V_2(t) = 0 \text{ for all } t.$$  

From (2.13) and (2.14) we have

$$A(t)V(t) = Q(t)R(t) \text{ for all } t,$$

and $R \in C^k(\mathbb{R}, \mathbb{F}^{n \times n})$ is of the form

$$R(t) = \begin{bmatrix} R_1(t) & 0 \\ 0 & 0 \end{bmatrix}$$

with $R_1(t) \in \mathbb{F}^{(n-r) \times (n-r)}$ upper triangular and full rank for all $t$.

Proof. Use Theorem 2.4 to get $U^*AV = \begin{bmatrix} S_1 \ 0 \end{bmatrix}$, and then use Lemma 2.1 to get

$$S_+(t) = Q_1(t)R_1(t).$$

Finally, let

$$Q(t) = U(t) \begin{bmatrix} Q_{11}(t) & 0 \\ 0 & I_{r \times r} \end{bmatrix}, \quad R(t) = \begin{bmatrix} R_1(t) & 0 \\ 0 & 0 \end{bmatrix}.$$
$U$ and $S_1$ are the first $n$ rows of $S$ in Theorem 2.4. Thus, it is enough to rewrite $A(t) = (U_1(t)V^*(t))(V(t)S_1(t)V^*(t)) = Q(t)P(t)$.

The next factorization is encountered in a number of applications; see [1, 12, 18] and see [4] for numerical study. The smoothness of its factors is proved similarly to how we proved Theorem 2.4.

**Theorem 2.7** (Takagi’s factorization). Let $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n \times n})$ be a complex symmetric matrix valued function (i.e., $A^T = A$) of constant rank: $\text{rank}(A(t)) \equiv n - r$ for all $t$ for fixed $r$ : $0 \leq r \leq n - 1$. Then there exists unitary $U \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n \times n})$ such that

\begin{equation}
(2.20) \quad A(t) = U(t) \begin{bmatrix} S_+ & 0 \\ 0 & 0 \end{bmatrix} U^T(t) \quad \text{for all } t,
\end{equation}

and $S_+ \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{(n-r)\times (n-r)})$ is symmetric positive definite.

Moreover, suppose that the continuous eigenvalues of $S_+$, $\lambda_1, \ldots, \lambda_{n-r}$, satisfy (2.5) for some $e \leq k$ and for all $t$ and $i \neq j$. Then there exists orthogonal $Q \in \mathcal{C}^{k-r}(\mathbb{R}, \mathbb{R}^{(n-r)\times (n-r)})$ such that $Q^TS_+Q = \text{diag}(\lambda_1, \ldots, \lambda_{n-r})$. The eigenvalues can be taken to be $C^k$ functions.

**Proof.** If $A$ is complex symmetric, then $A(t) = B(t) + iC(t)$, where $B, C \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{n \times n})$ are symmetric. Consider the symmetric function $M \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{2n \times 2n})$,

\begin{equation}
(2.21) \quad M(t) = \begin{bmatrix} B(t) & C(t) \\ C(t) & -B(t) \end{bmatrix},
\end{equation}

and notice that we have $\text{rank}(M) = 2(n - r)$ for all $t$; this fact follows from

\[ (1/2) \begin{bmatrix} I_n & -iI_n \\ -iI_n & I_n \end{bmatrix} M^*(t)M(t) \begin{bmatrix} I_n & -iI_n \\ -iI_n & I_n \end{bmatrix}^* = \begin{bmatrix} A^*(t)A(t) & 0 \\ 0 & A(t)A^*(t) \end{bmatrix}. \]

Thus, similarly to the proof of Theorem 2.4, we now obtain the $C^k$ block Schur decomposition of $M$:

\begin{equation}
(2.22) \quad W^T(t)M(t)W(t) = \begin{bmatrix} S_+(t) & 0 \\ 0 & -S_+(t) \end{bmatrix},
\end{equation}

where $S_+ \in \mathbb{R}^{(n-r)\times (n-r)}$ is symmetric positive definite, and $W$ is orthogonal of the form $W = \begin{bmatrix} W_1 & W_2 & Q_3 \end{bmatrix}$ with $W_1 = \begin{bmatrix} X \\ -Y \end{bmatrix}$ and $W_2 = \begin{bmatrix} Y \\ X \end{bmatrix}$. Now let

\begin{equation}
(2.23) \quad U_1(t) = X(t) - iY(t) \quad \text{for all } t,
\end{equation}

so that $U_1 \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n \times (n-r)})$ is orthonormal and

\begin{equation}
(2.24) \quad A(t) = U_1(t)S_+(t)U_1^T(t).
\end{equation}

Next, let $P_1(t) = U_1(t)U_1^T(t)$. Then $\text{rank}(P_1(t)) = n - r$ for all $t$, and from Theorem 2.3 there exists unitary $V \in \mathcal{C}^k(\mathbb{R}, \mathbb{C}^{n \times n})$ such that $V^*P_1(t)V(t) = \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix}$, with $P_1 \in \mathcal{C}^{(n-r)\times (n-r)}$, Hermitian. Write $V(t) = \begin{bmatrix} V_1(t) & V_2(t) \end{bmatrix}$, and let

\begin{equation}
(2.25) \quad U(t) = \begin{bmatrix} U_1(t) \\ V_2(t) \end{bmatrix}.
\end{equation}

Then, since $V_2^*P_1V_2 = (U_1^TV_2)^*(U_1^TV_2) = 0$, we get $V_2U_1 = 0$, and so $U(t)$ is unitary. Moreover, trivially

\[ A(t) = U(t) \begin{bmatrix} S_+(t) & 0 \\ 0 & 0 \end{bmatrix} U^T(t), \]
and (2.20) follows.

Finally, the statement about being able to eigendecompose $S_+$ under the assumption (2.5) is again a direct consequence of [6, Theorems 3.3 and 3.5].

We complete this section with a result on smoothness of the generalized SVD, which unfolds nicely as a consequence of things we proved earlier in this section. We first need the following elementary lemma.

**Lemma 2.8 (smooth Choleski factorization).** Let $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{n \times n})$ and $A(t)$ be a Hermitian (symmetric) positive definite function for all $t$. Then there exists a unique lower triangular function $G \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{n \times n})$, with positive diagonal entries, such that

\begin{equation}
A(t) = G(t)G^*(t) \quad \text{for all } t.
\end{equation}

**Proof.** Write $A = [ \begin{array}{cc} a_{11} & b \\ b^* & A \end{array} ]$. Let $G_1 = [ \begin{array}{cc} \sqrt{a_{11}} & 0 \\ b/\sqrt{a_{11}} & I_{n-1} \end{array} ]$. Clearly, $G_1 \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{n \times n})$, and $G_1^{-1}AG_1^{-*} = [ \begin{array}{cc} 1 & 0 \\ 0 & A \end{array} ]$, with $A_1 = A - bb^*/a_{11}$. Obviously, $A_1 \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{(n-1) \times (n-1)})$ and is positive definite, so repeating this procedure gives the result.

**Theorem 2.9 (generalized SVD).** Let $A \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{m \times n})$, $m \geq n$, be a constant rank function: $\text{rank}(A) = n - r$ for all $t$, $r : 0 \leq r \leq n - 1$, and let $B \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{p \times n})$, $p \geq n$, be full rank for all $t$. Then there exist unitary (orthogonal) $U_1 \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{m \times m})$, orthonormal $U_2 \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{n \times n})$, and invertible $X \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{n \times n})$ such that, for all $t$,

\begin{equation}
U_1^*(t)A(t)X(t) = \begin{bmatrix} S_A(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2^*(t)B(t)X(t) = \begin{bmatrix} S_B(t) & 0 \\ 0 & I_r \end{bmatrix},
\end{equation}

where $S_A \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{(n-r) \times (n-r)})$, $S_B \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{(p-r) \times (p-r)})$, $S_A$ is Hermitian positive definite, and

\[ S_A^*(t)S_A(t) + S_B^*(t)S_B(t) = I \quad \text{for all } t. \]

**Proof.** Clearly, $\text{rank}[ \begin{array}{c} A(t) \\ B(t) \end{array} ] = n$ for all $t$, and hence from Lemma 2.1 there exist orthonormal $Q \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{(n+r-p) \times n})$ and nonsingular upper triangular $R \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{n \times n})$ such that $[ \begin{array}{c} A(t) \\ B(t) \end{array} ] = Q(t)R(t) =: [ \begin{array}{c} Q_1(t) \\ Q_2(t) \end{array} ]$. Here, $Q_1 \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{m \times m})$ has constant rank $n - r$, and $Q_2$ has full rank $n$ for all $t$. Thus, from Theorem 2.4, there exist unitary (orthogonal) $U_1 \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{m \times m})$ and $V \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{n \times n})$ such that

\[ Q_1(t) = U_1(t) \begin{bmatrix} S_A(t) & 0 \\ 0 & 0 \end{bmatrix} V^*(t) \quad \text{for all } t, \]

and $S_A \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{(n-r) \times (n-r)})$ is Hermitian positive definite for all $t$.

Next, consider the following $\mathcal{C}^k$ orthonormal transformation

\begin{equation}
\begin{bmatrix} U_1^*(t) & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} V(t) = \begin{bmatrix} S_A(t) & 0 \\ 0 & 0 \end{bmatrix},
\end{equation}

where $W(t) = Q_2(t)V(t)$, and thus $W$ has full rank for all $t$. Partition $W(t) = (W_1(t) \ W_2(t))$, where $W_1 \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{p \times (n-r)})$, $W_2 \in \mathcal{C}^k(\mathbb{R}, \mathbb{F}^{p \times r})$. From (2.28),

\[ S_A^*(t)S_A(t) + W_1^*(t)W_1(t) = I, \quad W_1^*(t)W_2(t) = 0, \quad \text{and } W_2^*(t)W_2(t) = I \quad \text{for all } t. \]
Since \( W_1 \) is full rank for all \( t \), then \( W_1^*W_1 \) is positive definite. So, from Lemma 2.8, we have \( W_1^*W_1 = I - S_A^*S_A = GG^* \) for all \( t \), where \( G \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{(n-r)\times(n-r)}) \) is nonsingular and lower triangular. Now, we let

\[
U_2(t) = W(t) \begin{bmatrix} (G^*(t))^{-1} & 0 \\ 0 & I \end{bmatrix}, \quad S_B(t) = G^*(t) \quad \text{for all } t.
\]

It is easy to check that \( U_2(t) \) is orthonormal for all \( t \) and that

\[
Q_2(t) = U_2(t) \begin{bmatrix} S_B(t) & 0 \\ 0 & I \end{bmatrix} V^*(t).
\]

So, letting \( X(t) = (V^*(t)R(t))^{-1} \) for all \( t \), the result is proved.

3. Periodicity of the factors. In this section we consider the periodic case, \( A \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{m\times n}) \). Without loss of generality, we will take \( \tau = 1 \) and assume that 1 is the minimal period of \( A \). Also, we will henceforth assume that the function \( A \) does not have all constant eigenvalues. See Remarks 3.5 and 3.21 for what to expect in this case.

It is natural to inquire whether or not the (smooth) factors in section 2 (e.g., see Theorems 2.3 and 2.4) inherit some periodicity in case the function \( A \) is periodic of period 1. For example, can we say that \( Q \) in Theorem 2.3 has period 1? Besides being a question of theoretical interest, this inquiry has also practical implications, since it would indicate that it may be possible to compute a factorization of \( A \) over only one period. Guided by what we know from Floquet theory for differential equations and from the work of Sibuya in [19], we may expect that in the case \( F = \mathbb{R} \) the factors are periodic with twice the period they have in the case \( F = \mathbb{C} \).

We divide the results of this section in two parts. In the first part, we give somewhat “coarser” periodicity results: we look at periodicity of the factors for block eigendecompositions in the case in which the blocks correspond to disjoint groups of eigenvalues, and as a by-product we look at periodicity of the factorizations of section 2. In the second part, we look at the “finer” structure by allowing eigenvalues (singular values) to coalesce: for example, we ask ourselves about periodicity of the Schur factors in an eigendecomposition of a Hermitian matrix valued function.

3.1. Periodicity of block decompositions. The first group of results are lumped together in Theorem 3.3 below. The statements there are immediate consequences of similar results of section 2 and of Lemma 3.1 and Theorem 3.2, which extend Lemmas 2.1 and 2.8 and Theorem 2.3, to the periodic case.

**Lemma 3.1** (periodicity of \( QR \) and Choleski decompositions). We have the following.

- Let \( A \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{m\times n}) \), where \( m \geq n \), \( k \geq 0 \), and let \( A(t) \) be full rank for all \( t \). Then \( A \) admits a unique factorization \( A(t) = Q(t)R(t) \) for all \( t \), where \( Q \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{m\times n}) \) is orthonormal and \( R \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{n\times n}) \) is upper triangular with positive diagonal entries (real even in the case \( F = \mathbb{C} \)).
- Let \( A \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{n\times n}) \) and \( A \) be Hermitian positive definite. Then there exists a unique lower triangular function \( G \) with positive diagonal entries, \( G \in \mathcal{C}^k(\mathbb{R}, \mathbb{R}^{n\times n}) \), such that \( A(t) = G(t)G^*(t) \) for all \( t \).

**Proof.** Both for \( F = \mathbb{C} \) and \( F = \mathbb{R} \), the stated results about periodicity follow immediately from: (i) the Gram–Schmidt’s process on the columns of \( A \), for the \( QR \) factorization, and (ii) the procedure of the proof of Lemma 2.8, for the Choleski factorization. \( \square \)
Theorem 3.2 (Sibuya’s result and block Schur form). Let $A \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n \times n})$ with $k \geq 0$. Assume that $\Lambda(A) = \Lambda_1 \cup \cdots \cup \Lambda_l$, where $\Lambda_i \cap \Lambda_j = \emptyset$ for all $t$ and $i \neq j$, $i,j = 1, \ldots, l$. Then there exist invertible $S \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n \times n})$ and unitary $Q \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n \times n})$ such that

\begin{equation}
S^{-1}(t)A(t)S(t) = \begin{bmatrix}
E_{11} & 0 & \cdots & 0 \\
0 & E_{22} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & E_{ll}
\end{bmatrix} = E(t),
\end{equation}

\begin{equation}
Q^*(t)A(t)Q(t) = \begin{bmatrix}
T_{11} & T_{12} & \cdots & T_{1l} \\
0 & T_{22} & \cdots & T_{2l} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & T_{ll}
\end{bmatrix} = T(t),
\end{equation}

where each $T_{ij} \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n_i \times n_j})$ and $E_{ii} \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n_i \times n_i})$ with $\Lambda(E_{ii}) = \Lambda(T_{ii}) = \Lambda_i$, $i = 1, \ldots, l$, and $n_1 + \cdots + n_l = n$.

In case $\mathbb{F} = \mathbb{R}$, assume that det($\Lambda_i$) \(\in\) $\mathbb{R}$ for all $i$. Then the previous statements—in particular, (3.1) and (3.2)—are true for invertible $S \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n \times n})$ and orthogonal $Q \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n \times n})$. Now, each $T_{ij} \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n_i \times n_j})$ and $E_{ii} \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n_i \times n_i})$.

For either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, if $A$ is normal, then in (3.2) we have $T_{ij} = 0$, $i \neq j$, and $T_{ii}$, $i = 1, \ldots, l$, are also normal.

Proof. The smoothness results are in Theorem 2.3. As far as the periodicity results, we recall that in [19, Theorem 3, Remark 3], Sibuya proved the result (3.1) with $S$ of period 1 for two disjoint blocks of eigenvalues, in the case $\mathbb{F} = \mathbb{C}$. It is immediate to apply his result over and over to obtain (3.1) with $S$ of period 1 for $p$ disjoint blocks of eigenvalues. Application of Lemma 3.1 yields $S = QR$ and thus (3.2). In the real case, $\mathbb{F} = \mathbb{R}$, Sibuya (see [19, Remark 1]) gives (3.1) with $S$ of period 2 for two disjoint blocks of eigenvalues. One cannot simply apply this result over and over now, since this would increase the period of $S$. However, the arguments in Sibuya’s proofs can be readily generalized to dealing with $p$ blocks of eigenvalues simultaneously. So doing, one obtains $S \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n \times n})$ and then trivially $Q \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n \times n})$ in (3.2) applying Lemma 3.1 to get $S = QR$. The statement in the normal case is proved as in the last part of the proof of Theorem 2.3. □

Statements (i)–(v) in Theorem 3.3 below are extension to the periodic case of Theorem 2.4, Corollaries 2.5 and 2.6, and Theorems 2.7 and 2.9, respectively.

Theorem 3.3. Let $A \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{m \times n})$, respectively, $\mathbb{R}^{m \times n}$, $m \geq n$ and $k \geq 0$, be of constant rank: rank($A$) = $n - r$ for all $t$ and fixed $r$: $0 \leq r \leq n - 1$.

(i) There exist unitary $U \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{m \times m})$ and $V \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n \times n})$, respectively, orthogonal $U \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{m \times m})$ and $V \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n \times n})$, such that (2.4) holds, where $S_+ \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{(n-r) \times (n-r)})$, respectively, $S_+ \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{(n-r) \times (n-r)})$, is Hermitian, respectively, symmetric, positive definite.

(ii) There exists unitary $Q \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{m \times m})$ and $V \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n \times n})$, respectively, orthogonal $Q \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{m \times m})$ and $V \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n \times n})$, such that (2.17) and (2.18) hold with $R_1 \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{(n-r) \times (n-r)})$, respectively, $R_1 \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{(n-r) \times (n-r)})$.

(iii) There exist unitary $Q \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{m \times m})$, respectively, orthogonal $Q \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{m \times m})$, and Hermitian (symmetric) positive semidefinite $P \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n \times n})$, respectively, $P \in \mathcal{C}_t^2(\mathbb{R}, \mathbb{R}^{n \times n})$, such that (2.19) holds.

Now, let $A \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{m \times n})$ be complex symmetric with constant rank $n - r$, where $0 \leq r \leq n - 1$. Then,

(iv) there exists unitary $U \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{n \times n})$ and symmetric positive definite $S_+ \in \mathcal{C}_t^k(\mathbb{R}, \mathbb{C}^{(n-r) \times (n-r)})$ such that (2.20) holds.
Finally, let \( A \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{R}^{m \times n}) \), \( m \geq n \), be of constant rank \( n - r \), and let \( B \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{R}^{p \times n}) \), \( p \geq n \), be full rank for all \( t \). Then,

(v) there exist unitary \( U_1 \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{C}^{m \times m}) \) and orthonormal \( U_2 \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{C}^{p \times n}) \), respectively, orthogonal \( U_1 \in \mathcal{C}_2^k(\mathbb{R}, \mathbb{R}^{m \times m}) \) and \( U_2 \in \mathcal{C}_2^k(\mathbb{R}, \mathbb{R}^{p \times n}) \), and invertible \( X \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{C}^{n \times n}) \), respectively, \( X \in \mathcal{C}_2^k(\mathbb{R}, \mathbb{R}^{n \times n}) \), such that (2.27) holds.

We are now ready to provide sharper results in a number of cases. First, let us begin remarking that, as a consequence of Theorem 3.2, a function \( A \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{C}^{n \times n}) \) with distinct eigenvalues is diagonalizable with a \( \mathcal{C}_1^k \) function of eigenvectors (similarly, it has a \( \mathcal{C}_2^k \) Schur decomposition). This is because the diagonal blocks \( E_{ii} \) and \( T_{ii} \) in (3.1) and (3.2) are of period 1 when \( \mathbb{F} = \mathbb{C} \). However, in the real case \( \mathbb{F} = \mathbb{R} \), it appears that the blocks \( E_{ii} \) (and \( T_{ii} \)) are of period 2, since \( S \) and \( Q \) have period 2 in this case. Our next task is to show that, even in the real case, \( E_{ii} \) can be chosen of period 1, when \( A \) has only simple eigenvalues (real or complex conjugate), and that also \( T_{ii} \) can be chosen of period 1 in this case, when \( A \) is analytic.

**Theorem 3.4.** Let \( A \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{R}^{n \times n}) \), \( k \geq 0 \). Suppose that \( A(t) \) has only simple (real or complex conjugate) eigenvalues for all \( t \). Then, in (3.1), we can choose \( S \in \mathcal{C}_2^k(\mathbb{R}, \mathbb{R}^{n \times n}) \) such that \( E_{ii} \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{R}^{n \times n}) \) (here, \( n_i = 1 \) or \( 2 \)). Further, if \( A \) is normal the conclusions remain true with \( S \) orthogonal, i.e., in (3.2) we have \( T_{ii} \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{R}^{n \times n}) \).

**Proof.** Clearly, if \( A \) has only real distinct eigenvalues, then they have period 1, because they have period 1 seen as resulting from a complex similarity transformation. Next, consider the case of a complex conjugate pair, and write \( E_{ii}(t) = \begin{bmatrix} b_{11}(t) & b_{21}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} \), where each \( b_{ij} \) is a \( \mathcal{C}_2^k \) function. The eigenvalues of \( E_{ii}(t) \) are \( \lambda \) and \( \bar{\lambda} \), and they are both \( \mathcal{C}_2^k \) functions (since they are simple). Since \( E_{ii} \) is real, and the eigenvalues of \( A \) are distinct, then \( b_{12}(t) \neq 0 \) for all \( t \). Now, let \( x(t) = \frac{b_{22}(t) - b_{11}(t)}{2b_{12}(t)} \),

\[
a(t) = \frac{\text{Im}(\lambda(t))}{b_{12}(t)}, \quad S_1(t) = \begin{bmatrix} 1 \frac{1}{a(t)} \\ 0 1 \end{bmatrix}, \quad S_2(t) = \begin{bmatrix} \frac{1}{a(t)} & 0 \\ 0 & 1 \end{bmatrix},
\]

so that \( S_i \in \mathcal{C}_2^k(\mathbb{R}, \mathbb{R}^{2 \times 2}) \). We have

\[
(S_1(t)S_2(t))^{-1} E_{ii}(t) (S_1(t)S_2(t)) = \begin{bmatrix} \text{Re}(\lambda(t)) & \text{Im}(\lambda(t)) \\ -\text{Im}(\lambda(t)) & \text{Re}(\lambda(t)) \end{bmatrix} \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{R}^{2 \times 2}).
\]

Therefore, we can choose \( E_{ii} \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{R}^{2 \times 2}) \) even though \( S \in \mathcal{C}_2^k(\mathbb{R}, \mathbb{R}^{n \times n}) \).

The statement in the case of \( A \) normal is an immediate consequence of the fact that a \( (2 \times 2) \) nonsymmetric normal matrix has the form \( \begin{bmatrix} \text{Re}(t) & \text{Im}(t) \\ -\text{Im}(t) & \text{Re}(t) \end{bmatrix} \). \( \square \)

**Remark 3.5.** Of course, if \( A \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{R}^{n \times n}) \) with all eigenvalues constant, then Theorem 3.2 still holds. In this case, if all eigenvalues are distinct, \( E_{ii} \) and \( T_{ii} \) are constant.

### 3.2. Periodicity in coalescing case.

Our next task is to characterize the periodicity of the eigenvalues (singular values), and of the corresponding orthonormal factors, in the case in which eigenvalues are allowed to coalesce. We will restrict to the Hermitian case. The first step will be to establish periodicity of the eigenvalues, and then we will determine the periodicity of the unitary transformation.

So, we have a function \( A \in \mathcal{C}_1^k(\mathbb{R}, \mathbb{C}^{n \times n}) \), \( A = A^* \), or also \( A \) analytic: \( A \in \mathcal{C}_2^k \), and we want to establish the periodicity of its eigenvalues. This seemingly simple problem is rather complex, since there is a very delicate interplay between smoothness of the eigenvalues and their periodicity. Let us begin with the following observation.

- The eigenvalues of \( A \) are roots of the characteristic polynomial of \( A \), call it \( \pi_1(\lambda, t) \), and clearly \( \pi_1(\lambda, t+1) = \pi_1(\lambda, t) \). Therefore, we can certainly label the eigenvalues of \( A \) so that they are periodic functions of period 1. Since the eigenvalues are continuous functions, we thus immediately have that there
exists an ordering of the eigenvalues so that they are (at least) $C^0$ functions. Now, if the eigenvalues are simple, then we know that they can be taken as $C^1$ functions (this follows from Theorem 3.2). However, if two eigenvalues coalesce and we want them to be periodic functions of period 1, then in general we may have to settle for $C^0$ eigenvalues. Alternatively, we must be willing to increase the period in order to retain smoothness. This fundamental conflict between having period 1 and/or maximal possible smoothness is already present in the analytic case. Recall that if $A = A^* \in C^\omega$, then it has an analytic diagonalization $A = QDQ^*$ regardless of whether or not the eigenvalues coalesce; see [14].

Example 3.6. Consider the symmetric function $A(t) = \begin{bmatrix} 1 - \frac{1}{4}\sin^2 \pi t & -\frac{1}{8}\sin 2\pi t \\ -\frac{1}{8}\sin 2\pi t & \frac{1}{4}\sin^2 \pi t \end{bmatrix}$, $t \in \mathbb{R}$. Clearly, $A \in C^\omega$, and thus we know that it has analytic eigenvalues (and eigenvectors). The analytic eigenvalues are $\lambda_1(t) = \frac{1 + \cos \pi t}{2}$, and $\lambda_2(t) = \frac{1 - \cos \pi t}{2}$, and we notice that $\lambda_1 = \lambda_2$ at $t = 1/2, 3/2, \ldots$. The associated orthogonal function of eigenvectors is $Q = \begin{bmatrix} \cos \frac{\pi}{2}t & \sin \frac{\pi}{2}t \\ -\sin \frac{\pi}{2}t & \cos \frac{\pi}{2}t \end{bmatrix}$, so that $Q^T(t)A(t)Q(t) = \begin{bmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{bmatrix}$. Notice that $\lambda_{1,2} \in C^\omega_2$ and $Q \in C^\omega_4$; in other words, we retained analyticity of the eigenvalues, but we have doubled their period (with respect to that of $A$) and then doubled again the period of $Q$. Of course, we could have chosen the eigenvalues to be merely continuous, and of period 1, as follows:

$$\tilde{\lambda}_1(t) = \begin{cases} \frac{1 + \cos \pi t}{2} & \text{if } 0 \leq t \leq 1/2 \text{ or } 3/2 \leq t \leq 2, \\ \frac{1 - \cos \pi t}{2} & \text{if } 1/2 \leq t \leq 3/2, \end{cases}$$

$$\tilde{\lambda}_2(t) = \begin{cases} \frac{1 - \cos \pi t}{2} & \text{if } 0 \leq t \leq 1/2 \text{ or } 3/2 \leq t \leq 2, \\ \frac{1 + \cos \pi t}{2} & \text{if } 1/2 \leq t \leq 3/2. \end{cases}$$

Figure 1 summarizes the situation: the curves on the top, bottom, of the value 1/2 give $\tilde{\lambda}_{1,2}$.

Example 3.7. For all $t$, take $A(t) = Q(t)D(t)Q^T(t)$, with $D(t) = \text{diag}(\lambda_1(t), \lambda_2(t))$, $\lambda_1(t) = \cos^2 \pi t$, $\lambda_2(t) = \frac{1}{2} \sin^2 \pi t$, and $Q = \begin{bmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{bmatrix}$. Easily, $A, D \in C^\omega$, while $Q$ is analytic of period 2. Figure 2 exemplifies the situation: even though the functions

![Figure 1](image-url)
Based upon Examples 3.6 and 3.7, we conjecture that the periodicity of the eigenvalues depend on the relative positioning of coalescing eigenvalues after one period. This conjecture turns out to be essentially correct, as we set out to prove.

Consider the general case $A \in \mathbb{C}^{k,\omega}_1$ or $A \in \mathbb{C}^{k,\omega}_2$; for short, we will write this as $A \in \mathbb{C}^{k,\omega}$. Of course, we can (and will) think that the problem has been reduced to a block diagonal form as in (2.2), with the $T_{ii}$ being associated to disjoint groups of eigenvalues. To be precise, we will assume that the $T_{ii}$ satisfy Assumptions 3.8(i–ii) below.

**Assumption 3.8.** Let $B = B^* \in \mathbb{C}^{k,\omega}_1$ be such that

(i) “$B$’s eigenstructure is as fine as possible.” By this we mean that there do not exist $\Lambda^{(1)}$ and $\Lambda^{(2)}$, both nonempty, such that $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$ and $\Lambda^{(1)} \cup \Lambda^{(2)} = \Lambda(B)$ for all $t$.

(ii) “If $B \notin \mathbb{C}^\omega$, its eigenvalues satisfy condition (2.5).”

Assumption 3.8(ii) relative to each $T_{ii}$ in (2.2) implies (see [6]) that each $T_{ii}$ can be diagonalized with unitary (orthogonal) $U_{ii} \in \mathbb{C}^{k-\epsilon_i}(\mathbb{R}, \mathbb{F}^{n_i \times n_i})$. Therefore, if we let $e = \max e_i$, $i = 1, \ldots, l$, in the $\mathbb{C}^k$ case, there exists a unitary matrix $Q \in \mathbb{C}^{k-\epsilon}(\mathbb{R}, \mathbb{F}^{n \times n})$ which diagonalizes $A$:

$$Q^* A Q =: D = \text{diag}(D_1, D_2, \ldots, D_l), \quad D_i = \text{diag}(\lambda_{ij}^{(i)}, j = 1, \ldots, n_i).$$

Our goal is to establish the periodicity of $D$, and the first step will be to establish the periodicity of the $D_i$’s. We need the following definition.

**Definition 3.9.** Given an (integer) partition of $n$: $n_1, \ldots, n_l$ so that $n_1 + \cdots + n_l = n$, let $\mu(n) := \text{lcm}(n_1, \ldots, n_l)$ (lcm is the least common multiple). For given value of $n$, let $\mu^*(n)$ be the maximum value of $\mu(n)$ over all partitions of $n$.

**Theorem 3.10** (periods of eigenvalues). Let $A \in \mathbb{C}^{k,\omega}_1(\mathbb{R}, \mathbb{F}^{n \times n})$, $A = A^*$, with eigenvalues not all constant. Let the function $A$ be in the form (2.2), where the diagonal blocks satisfy Assumption 3.8(i)–(ii). Then, with the notation of (3.3), the
functions $D_i \in C^{k,\omega}$ can be taken periodic of periods $p_i$, with $1 \leq p_i \leq n_i$.\footnote{The precise value of $p_i$ depends on the eigenvalues' relative positions after one unit of time; see Lemma 3.14} If $D_i = \lambda_i I$, $\lambda_i$ constant, then we can take $p_i = 1$. The diagonal function of eigenvalues, $D$, can be taken periodic of period $p$, the least common multiple of the $D_i$’s periods, $1 \leq p \leq \mu^*(n)$.

Proof. The result follows from Lemma 3.14 below. \qed

Remark 3.11. It should be appreciated that the function $\mu^* : \mathbb{N} \to \mathbb{N}$ grows very rapidly. Although a closed formula for the exact values of $\mu^*(n)$ appears hard to obtain,\footnote{Our colleague Yang Wang communicated to us the asymptotic $\log(\mu^*(n)) \approx \sqrt{n \log n}$.} it is easy to obtain the upper bound

\begin{equation}
\log(\mu^*(n)) \leq \frac{n}{e}.
\end{equation}

The verification of (3.4) is simple, since

$$\text{lcm}(n_1, n_2, \ldots, n_l) \leq \left(\frac{n_1 + n_2 + \cdots + n_l}{l}\right)^l = \left(\frac{n}{l}\right)^l$$

for all $l$,

and $f(x) = (\frac{x}{n})^e$ has a global maximum at $x_0 = \frac{n}{e}$. (In comparison, $\mu^*(10) = 30$ and $e^{\frac{10}{e}} \approx 40$, but already $\mu^*(16) = 140$ and $e^{\frac{16}{e}} \approx 360$.)

To prove Theorem 3.10, we need the following lemmas. We will use the concept of irreducible matrix (e.g., see [13]). Also, given a constant matrix $B$, we will call period of $B$ the smallest integer $k \geq 1$, if it exists, such that $B^k = I$.

Lemma 3.12. If $P$ is an $(n, n)$ irreducible permutation matrix, then it has period $n$.

Proof. From [13, p. 512], an irreducible permutation matrix $P$ is similar, via a permutation similarity $\Pi$, to a cyclic matrix

$$\Pi P \Pi^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and thus $P^n = I$, but $P^k \neq I$ if $k < n$. \qed

Lemma 3.13. If $P$ is an $(n, n)$ permutation matrix, then it has the irreducible decomposition $\text{diag}(P_{ii}, i = 1, \ldots, l)$, with $P_{ii}$ an $(n_i, n_i)$ irreducible permutation, $i = 1, \ldots, l$. The period of $P$ is $\mu(n)$, where the partition of $n$ is that associated to its irreducible decomposition.

Proof. If $P$ is irreducible, Lemma 3.12 gives the result. So, let $P$ be reducible. Then there exists a permutation matrix $\Pi$ such that $\Pi P \Pi^T = \begin{bmatrix} P_{11} & P_{12} \\ \vdots & \vdots \\ P_{l1} & P_{l2} \end{bmatrix}$. Since $\Pi$ is a permutation, then $P_{22}$ is necessarily also a permutation and hence $P_{12} = 0$. We continue this reduction process until all diagonal blocks are irreducible, and apply Lemma 3.12. \qed

Lemma 3.14 (minimal integer period). Under the assumptions of Theorem 3.10, there exists a smallest integer $p$ such that $D(t+p) = D(t)$ for all $t$. We call this value $p$ the minimal integer period of $D$, and we have $1 \leq p \leq \mu^*(n)$.

Proof. First consider the $C^k$ case. By assumption, the points where eigenvalues coalesce are isolated, and thus there is only a finite number of points in $[0, 1]$ where eigenvalues coalesce. Without loss of generality, let the eigenvalues be distinct at $t = 0$, and let $0 < t_1 < \cdots < t_{M+1} < 1$ be the points in $[0, 1]$ where some eigenvalues coalesce.
Let $I_j = (t_j, t_{j+1})$, $j = 1, \ldots, M$, and $I_0 = (t_{M+1}, t_1)$, so that the eigenvalues are distinct on each of these subintervals. Let $D^{(j)}(t) = \text{diag}(\lambda_{j_1}(t), \ldots, \lambda_{j_n}(t))$ be a labeling of the eigenvalues which reflects their relative ordering on each $I_j$; that is, for all $t \in I_j$, $\lambda_{j_1}(t) > \cdots > \lambda_{j_n}(t)$; let $D^{(0)}$ be called simply $D$. Let $P_j$ be the permutation matrix defined so that

$$P_j D^{(j-1)}(t) P_j^T = D^{(j)}(t), \quad j = 1, \ldots, M + 1, \quad t \in I_j.$$

In particular, if we take a point $\tau_0 \in I_0$, then we have

$$P_{M+1} D^{(M)}(1 + \tau_0) P_{M+1}^T = D^{(M+1)}(1 + \tau_0),$$

and thus also

$$D^{(M+1)}(1 + \tau_0) = P_{M+1} P_M D^{(M-1)}(1 + \tau_0) P_M^T P_{M+1}^T = \cdots = P_{M+1} \cdots P_1 D(1 + \tau_0) P_1^T \cdots P_{M+1}^T.$$

Since the roots of the characteristic polynomial have period 1 (as locus of points), then $D^{(M+1)}(1 + \tau_0) = D(\tau_0)$, and if we let $P_1 := P_{M+1} \cdots P_1$, we then have

$$D(\tau_0) = P_1 D(1 + \tau_0) P_1^T \quad \text{and further} \quad D(\tau_0) = P_1^l D(l + \tau_0) (P_1^l)^l, \quad l = 1, 2, \ldots.$$

Now, let $p_1$ be the smallest integer such that $P_1^{p_1} = I$. Then, we obtain that $D(t + p_1) = D(t)$ for all $t \in I_0$. In precisely the same way, we now take a point $\tau_j \in I_j$ and repeat the reasoning to eventually obtain that there are permutation matrices $P_2, P_3, \ldots, P_{M+1}$ such that

$$D^{(j)}(\tau_j) = P_{j+1} D^{(j)}(1 + \tau_j) P_{j+1}^T, \quad j = 1, \ldots, M,$$

and hence if $p_j$ are the periods of these $P_j$, we would have that $D$ itself has periods $p_j$ for all $t \in I_j$. But it is a simple observation that all $p_j$ are equal; e.g., it is immediate that $P_2 = P_1 P_1^T$, etc. We let $p$ be this common value, and so we have obtained that $D(t + p) = D(t)$ for all $t \in \bigcup_j I_j$. Finally, because of continuity, the function $D$ has period $p$ everywhere. Applying Lemmas 3.12 and 3.13 we get the bound on $p$: $1 \leq p \leq \mu^*(n)$.

In the $C^\omega$ case, we have $T_{it} \in C^\omega$ and $D_t \in C^\omega$, but we may now have that some eigenvalues are identical for all $t$. Let $\mu_1, \ldots, \mu_m$ ($m \leq n$) be the eigenvalues of $A$, so that no two of them are identical for all $t$. Since the $\mu_i$'s are real analytic functions, then these functions must have a finite order of contact; that is, there exists an integer $e < \infty$ such that $\mu_i^{(e)}(t) \neq \mu_j^{(e)}(t)$ for all $t$ and $i \neq j$. Thus, the points where the $\mu_i$'s coalesce are isolated. We now can repeat the reasoning of the $C^k$ case relative to $M = \text{diag}(\mu_1, \ldots, \mu_m)$. \hfill \Box

We also have the following lemma.

**Lemma 3.15.** Under the assumptions of Theorem 3.10, $D$ cannot have irrational period.

**Proof.** Suppose there exists irrational $b$ such that $D(t+b) = D(t)$ for all $t$. Because of Lemma 3.14, there exists a minimal integer period $p$ such that $D(t+p) = D(t)$. Then there exist integers $k_1, k_2$ such that $\lambda_j(t + k_1 b) = \lambda_j(t + k_2 p)$ for all $t$ and $j = 1, \ldots, n$. But then all $\lambda_j$ must be constant, a case which we have excluded. \hfill \Box

Up to this point, we know that the function $D$ has minimal integer period $p$, $1 \leq p \leq \mu^*(n)$. However, we have left open the possibility for $D$ to have minimal
period given by a rational number \( p/q \), with \( (p, q) = 1 \) (relatively prime) and \( p \) the minimal integer period. Indeed, we now show—constructively—that there exist Hermitian functions \( B \) of period 1 with \( D \) having such values \( p/q \) as periods.

Let us begin by constructing functions \( B = B^* \in C^{k,\omega}_1(\mathbb{R}, \mathbb{C}^{m \times m}) \) such that \( Q^*(t)B(t)Q(t) = D(t) \) for all \( t \) with \( D \in C^{k,\omega}_{m/q}(\mathbb{R}, \mathbb{R}^{m \times m}) \) diagonal, \( (m, q) = 1 \), and \( Q \in C^\omega_m(\mathbb{R}, \mathbb{C}^{m \times m}) \) unitary.

**Lemma 3.16.** There exist \( D \in C^{k,\omega}_{m/q}(\mathbb{R}, \mathbb{R}^{m \times m}) \) diagonal and unitary \( Q \in C^\omega_m(\mathbb{R}, \mathbb{C}^{m \times m}) \) such that by letting \( B(t) := Q(t)D(t)Q^*(t) \) for all \( t \), then \( B \in C^{k,\omega}_1(\mathbb{R}, \mathbb{C}^{m \times m}) \). Further, \( B \) satisfies Assumption 3.8(i)–(ii).

**Proof.** We are going to take \( D(t) = \text{diag}(\lambda_1(t), \ldots, \lambda_m(t)) \) of appropriate smoothness, and of period \( m/q \), such that

\[
\lambda_j(t + 1) = \lambda_{j+1}(t) \quad \text{for } j = 1, \ldots, m \pmod{m}.
\]

Specifically, for \( j = 1, \ldots, m \) and for all \( t \), we take

\[
\lambda_j(t) = \begin{cases} 
\cos(\frac{2\pi}{m}j(t+j-1)) & \text{in the } C^\omega \text{ case,} \\
\cos^{k+1}(\frac{2\pi}{m}j(t+j-1)) & \text{in the } C^k \text{ case.}
\end{cases}
\]

In either case, for such \( D \) we have

\[
D(t + 1) = PD(t)P^T \text{ for all } t, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & \cdots & 0 \end{bmatrix}.
\]

Now we want to find \( Q \) of period \( m \) such that \( B \) has period 1.

**Claim 3.17.** The following two properties hold.

(i) If \( Q(t+1) = Q(t) \) for all \( t \), and \( P \) in (3.7), then \( B \) has period 1.

(ii) If we let \( \alpha = \frac{2\pi}{m} \) and \( \alpha_k = k\alpha, k = 1, 2, \ldots, m-1, \) then

\[
\sum_{j=1}^m e^{ij\alpha_k} = 0 \text{ for all } k = 1, \ldots, m-1.
\]

**Verification of Claim 3.17.** To verify (i), observe that if \( Q(t+1) = Q(t) \) then \( B(t+1) = Q(t+1)D(t+1)Q^*(t+1) = B(t) \) for all \( t \). To verify (ii), we let \( z = e^{i\alpha_k} \), so that \( z \neq 1 \), but \( z^m = 1 \). Thus, (ii) follows, since \( 0 = z^m - 1 = (z-1)(z^{m-1} + \cdots + z + 1) \).

We are now ready to define \( Q \). We take

\[
Q(t) := \begin{bmatrix} q_1(t) & q_1(t+1) & \cdots & q_1(t+m-1) \end{bmatrix}, \quad q_1(t) = \frac{1}{\sqrt{m}} \begin{bmatrix} e^{i\frac{2\pi}{m}t} \\ \vdots \\ e^{i\frac{(m-1)2\pi}{m}t} \\ e^{i\frac{2\pi}{m}t} \end{bmatrix}.
\]

In (3.8), clearly \( q_1 \in C^\omega_m \) and so does \( Q \), and \( m \) is the minimal period of \( Q \). Moreover, by construction, \( Q(t+1)P = Q(t) \). Direct verification, using Claim 3.17(ii), gives \( Q^*(t)Q(t) = I \) for all \( t \). Finally, using Claim 3.17(i), the lemma is proved.

**Remark 3.18.** With the help of Lemma 3.16, we can build matrices \( B \in C^{k,\omega}_1(\mathbb{R}, \mathbb{C}^{m \times m}) \) satisfying Assumption 3.8(i)–(ii), \( B = QDQ^* \), \( D = \text{diag}(\lambda_1, \ldots, \lambda_m) \), \( D \in C^{k,\omega}_{p/q}, p = 1, \ldots, m, \) and \( Q \in C^\omega_p \) unitary. In fact, we may let \( D = \begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix} \) with
\( D_1 = \text{diag}(\lambda_1, \ldots, \lambda_p) \) defined as \( D \) was in the proof of Lemma 3.16, and \( D_2 = \alpha I_{n-p} \) for appropriate constant \( \alpha \) chosen so that we satisfy Assumption 3.8(i)–(ii). Accordingly, let \( Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 1 \end{bmatrix} \) with \( Q_1 \) defined (relative to \( D_1 \)) as \( Q \) was relative to \( D \) in the proof of Lemma 3.16.

**Remark 3.19.** As a consequence of Remark 3.18, we can justify the claim that there exist \( B \in \mathbb{C}_1^{n^\omega} \), satisfying the assumptions of Theorem 3.10 with \( D \in \mathbb{C}_p^{n^\omega} \) and \( p : 1 \leq p \leq \mu^*(n) \). To achieve this in the \( \mathbb{C}^\omega \) case, take \( D = \text{diag}(D_1, \ldots, D_l) \) with \( D_j \in \mathbb{C}_p^{n_j} \) diagonal of dimension \( n_j \), \( 1 \leq p_j \leq n_j \), according to Remark 3.18, and build \( Q \) in a similar block diagonal fashion. In the \( \mathbb{C}^k \) case, in order to ensure that Assumption 3.8(ii) is satisfied, we may have to shift the spectra of some of the \( D_j \) appropriately.

Next, we examine the period of the eigenvectors \( Q \) in (3.3). Partition \( Q = \begin{bmatrix} Q_1 & Q_2 & \ldots & Q_l \end{bmatrix} \) conformally to \( D \)'s partitioning, so that we have

\[
AQ_i = Q_i D_i, \quad i = 1, \ldots, l, \quad D_i = \text{diag}(\lambda_j^{(i)}, j = 1, \ldots, n_i).
\]

So far, we have established that each \( D_i \) has period \( p_i/q_i \), where \( 1 \leq p_i \leq n_i \), \((p_i, q_i) = 1\).

The denominators \( q_i \) in the periods \( p_i/q_i \) of the functions \( D_i \) play no further role in what follows, and we will thus dispense with them, simply working with \( p_i \)-periodic \( D_i \)'s. This said, it is worth stressing once more that the minimal period of \( D_i \) may be the rational number \( p_i/q_i \); thus, also the minimal period of the (smooth) eigenvalues of \( A \) may well be a rational number \( p/q \). \((p, q) = 1, 1 \leq p \leq \mu^*(n) \). This same observation holds true also for the singular values of Theorems 3.24 and 3.27.

**Remark 3.20.** In [9], Gingold and Hsieh devised a Schur decomposition procedure for an analytic matrix-valued function \( A \) with real and analytic eigenvalues, which in particular is valid for a Hermitian analytic function. Then, in [9, Theorem 10.1], they noticed that if \( A \) and its analytic eigenvalues both have period 1, then their procedure will produce analytic unitary factors also of period 1. Clearly, in light of our results, one cannot generally assume that the (analytic) eigenvalues have period 1. However, we notice that it is enough to replace Gingold's and Hsieh's assumption of eigenvalues of period 1 with that of "eigenvalues" matrix \( D \) of minimal integer period \( p_i \) and then the procedure of Gingold and Hsieh delivers a unitary, \( p \)-periodic, analytic, matrix-valued function of eigenvectors. The validity of our observation is immediately verified upon examining the procedure of [9].

**Remark 3.21.** If \( A \in \mathbb{C}_1^k \) has all constant eigenvalues, then because of Assumption 3.8(ii) they must be distinct, and thus Theorem 3.2 applies. If instead \( A \in \mathbb{C}_1^k \), then even if all eigenvalues of \( A \) are constant, and possibly many of them identical, the procedure of [9] still delivers a unitary and analytic \( Q \) of period 1.

Because of Remark 3.20, in the \( \mathbb{C}^\omega \) case we can take the unitary eigenvectors \( Q \in \mathbb{C}^\omega \) with period given by the \( \text{lcm}(p_1, \ldots, p_l) \). The next result is for the \( \mathbb{C}^k \) case.

**Proposition 3.22.** Let \( A = A^* \in \mathbb{C}_1^k(\mathbb{R}, \mathbb{C}^{n \times n}) \), or \( A = A^T \in \mathbb{C}_1^k(\mathbb{R}, \mathbb{R}^{n \times n}) \), be such that its eigenvalues satisfy condition (2.5). Let \( D_1, \ldots, D_l \) be diagonal matrix-valued functions, \( D_i \cap D_j = \emptyset, i \neq j \), grouping the eigenvalues of \( A \), as fine as possible according to Assumption 3.8(i); thus, we can take \( D_i \in \mathbb{C}_p^{n_i}(\mathbb{R}, \mathbb{R}^{n_i \times n_i}), 1 \leq p_i \leq n_i, i = 1, \ldots, l \). Let unitary \( Q \in \mathbb{C}_p^{k-\varepsilon}, Q = \begin{bmatrix} Q_1 & Q_2 & \ldots & Q_l \end{bmatrix} \), be such that \( AQ_i = Q_i D_i, i = 1, \ldots, l \). Then, we can take each \( Q_i \) of period \( p_i \) and hence \( Q \) of period given by \( \text{lcm}(p_1, p_2, \ldots, p_l) \).

If the function \( A \) is symmetric real valued, and \( Q_i \) are real orthonormal, then each \( Q_i \) can be taken of period 2\( p_i \) and \( Q \) of period 2\( \text{lcm}(p_1, p_2, \ldots, p_l) \).
The proof of Proposition 3.22 follows from Theorem 3.23 below and Lemma 2.2.

**Theorem 3.23.** Let \( A = A^* \), \( A \in C^k([0, \infty)) \). Suppose that \( \Lambda(A) = \Lambda_1 \cup \Lambda_2 \) and \( \Lambda_1 \cap \Lambda_2 = \emptyset \) for all \( t \). Let unitary \( Q \in C^k \) be such that \( Q^*AQ = [T_{ij}]_1 \), \( \Lambda(T_{ii}) = \Lambda_1 \), \( i = 1, 2 \), and suppose that \( T_{11} \) satisfies Assumption 3.8. With \( \epsilon_1 \) given in (2.5), let \( Q_1 \in C^{k-\epsilon_1}([0, \infty)) \) such that \( Q_1^*T_1Q_1 = D_1 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) be such that \( Q_1^*Q_1 = 1 \in C_k([0, \infty)) \), \( 1 \leq \epsilon_1 \leq n_1 \). Let \( Q \) be \( [Q_{11} \ 0] \) such that (3.10) holds.

\[
AQ_1 - Q_1D_1 = 0 \quad \text{for all } t.
\]

Then there exists orthonormal \( \tilde{Q}_1 \in C^{k-\epsilon_1}([0, \infty)) \) such that (3.10) holds. If \( A \) and \( \tilde{Q}_1 \) are real valued, then \( \tilde{Q}_1 \) can be chosen of period 2\( p_1 \).

**Proof.** The basic idea of the proof, motivated by [19, Theorem 6], consists of modifying the function \( Q_1 \) of (3.10) to bring it into a periodic orthonormal one still satisfying (3.10). Notice that our assumption implies that the points where the eigenvalues coalesce are isolated; therefore without loss of generality we can assume that the eigenvalues are distinct at \( t = 0 \). Also, in what follows we let \( l = k - \epsilon_1 \).

First, observe that if \( \pi(\lambda, t) \) is the characteristic polynomial of \( A(t) \), then \( \pi(\lambda, t) = \pi_1(\lambda, t)\pi_2(\lambda, t) \), where \( \pi_1(\lambda, t) := (\lambda - \lambda_1(t)) \cdot (\lambda - \lambda_n(t)) \) and \( \pi_1(\cdot, t) \) and \( \pi_2(\cdot, t) \) do not have common roots. Therefore,

\[
\text{rank } \pi_1(A(t), t) = n - n_1 \quad \text{and} \quad \text{rank } \pi_2(A(t), t) = n_1, \quad \text{for all } t.
\]

Next, we observe that

\[
\pi_1(A(t), t)Q_1(t) = 0 \quad \text{for all } t, \quad \text{where } Q_1 \text{ satisfies (3.10)}.
\]

To show (3.11) is a simple computation:

\[
\pi_1(A, t)Q_1 = (A - \lambda_1 I) \cdots (A(t)Q_1 - \lambda_1 Q_1)
= (A - \lambda_1 I) \cdots (A - \lambda_{n_1} I)Q_1(D - \lambda_{n_1} I) = \cdots
= Q_1(D - \lambda_1 I)(D - \lambda_2 I) \cdots (D - \lambda_{n_1} I)
= Q_1 \text{diag}(0, \lambda_2, \ldots, \lambda_{n_1}) \cdots \text{diag}(\lambda_1, \ldots, \lambda_{n_1 - 1}, 0) = 0 \quad \text{for all } t.
\]

Therefore, \( Q_1(\cdot + p_1) \) and \( Q_1(\cdot) \) satisfy the same linear system (3.11), which has constant rank \( n - n_1 \) and their ranks are equal to \( n_1 \) for all \( t \). In particular, this implies that there exists a sufficiently small \( \rho > 0 \) and \( C \in C^l([-\rho, \rho], C^{n_1 \times n_1}) \) such that

\[
Q_1(t + p_1)C(t) = Q_1(t), \quad |t| \leq \rho.
\]

Since both \( Q_1(t + p_1) \) and \( Q_1(t) \) are orthonormal, clearly \( C^*C = I \). Since the eigenvalues are distinct at \( t = 0 \), then, for \( |t| \leq \rho \), \( C(t) = \text{diag}(e^{i\phi_j(t)}, j = 1, \ldots, n_1) \) with \( \phi_j \in C^l \). Next, let \( \tilde{C}(t) = C^*(0)C(t) \) for all \( t \in [-\rho, \rho] \), so that \( \tilde{C}(0) = I \) and \( [Q_1(t + p_1)C(0)]\tilde{C}(t) = Q_1(t) \). Further, for \( \rho \) sufficiently small, the following function is well defined:

\[
R(t) = \frac{1}{2}(I + \tilde{C}(t))^{-1}(I - \tilde{C}(t)), \quad |t| \leq \rho,
\]

and notice that \( R^*(t) = -R(t) \) and \( R(0) = 0 \). Now take a function \( v \) which has continuous derivatives of all orders, \( 0 \leq v(t) \leq 1 \) for all \( t \), \( v(t) = 1 \) for \( t \geq 0 \) and
$v(t) = 0$ for $t \leq -r_0$, where $r_0 > 0$ is sufficiently small, $r_0 \leq \rho$ (such $v$ is called a mollifier in [8]). Then define

$$\tilde{R}(t) = v(t)R(t), \quad -r_0 \leq t \leq 0,$$

and notice that $\tilde{R}(t) = 0$ for $-\infty < t \leq -r_0$. Let

$$\hat{C}(t) = \begin{cases} C(t), & 0 \leq t \leq \rho, \\
C(0)(I - 2\tilde{R}(t))(I + 2\tilde{R}(t))^{-1}, & t \leq 0,
\end{cases}$$

notice that $\hat{C} \in \mathcal{C}^l$ is unitary (and diagonal), and set

$$\hat{Q}_1(t) = Q_1(t)\hat{C}(t - p_1) \text{ for all } t \leq p_1 + \rho.$$

Thus, $\hat{Q}_1$ is orthonormal, $\mathcal{C}^l$ on $[-\rho, p_1 + \rho]$, and satisfies (3.10). Using (3.12), we obtain

$$\hat{Q}_1(t) = \begin{cases} Q_1(t)C(0), & -\rho \leq t \leq p_1 - r_0, \\
Q_1(t - p_1), & p_1 \leq t \leq p_1 + \rho.
\end{cases} \quad (3.13)$$

Now, take a $C^\infty(\mathbb{R})$ function $w$ such that $0 \leq w(t) \leq 1$ for all $t$, $w(t) = 1$ for $t \geq p_1 - r_1$ and $w(t) = 0$ for $t \leq r_1$, where $r_1 > 0$ is sufficiently small, $r_1 \leq r_0$. Let

$$L = \log C(0), \quad N(t) = w(t)L, \quad H(t) = \exp(N(t)) \text{ for all } t,$$

so that $L^* = -L$, $H$ is $C^\infty$ and unitary for all $t$, and

$$H(t) = \begin{cases} I_{n-m_1}, & t \leq r_1, \\
C(0), & t \geq p_1 - r_1.
\end{cases}$$

Finally, let

$$\hat{Q}_1(t) = \hat{Q}_1(t)H(t), \quad -r_1 \leq t \leq p_1 + r_1, \quad (3.14)$$

so that $\hat{Q}_1 \in \mathcal{C}^l([-r_1, p_1 + r_1], \mathbb{C}^{n \times m_1})$, and $\hat{Q}_1$ is orthonormal and satisfies (3.10). Moreover, $\hat{Q}_1$ satisfies

$$\hat{Q}_1(t) = \begin{cases} Q_1(t)C(0), & -r_1 \leq t \leq r_1, \\
Q_1(t - p_1)C(0), & p_1 \leq t \leq p_1 + r_1.
\end{cases}$$

In particular, $\hat{Q}_1(t + p_1) = \hat{Q}_1(t)$, $0 \leq t \leq r_1$, and thus the proof of the theorem follows by periodically extending $\hat{Q}_1([0, p_1], \mathbb{C}^{n \times m_1})$.

In the case where $A$ is real valued, and $Q_1$ and thus $C$ in (3.12) are real as well, the previous construction fails because $\log(C(0))$, and hence $H(t)$ are complex valued, in general. This is because now $C(0)$ is a diagonal matrix of $\pm 1$. To remedy this, define the function $\tilde{Q}_1$ by

$$\tilde{Q}_1(t) = \begin{cases} \tilde{Q}_1(t + p_1)C(0), & -p_1 - \rho \leq t \leq 0, \\
\tilde{Q}_1(t), & 0 \leq t \leq p_1 + \rho.
\end{cases}$$
Then $\bar{Q}_1$ is orthonormal, $\bar{Q}_1 \in C^l([-p_1 - \rho, p_1 + \rho], \mathbb{R}^{n \times m_1})$, and satisfies (3.10).

Furthermore,

$$\bar{Q}_1(t) = \begin{cases} Q_1(t + p_1)(C(0))^2, & -p_1 - \rho \leq t \leq -r_0, \\ Q_1(t - p_1), & p_1 \leq t \leq p_1 + \rho. \end{cases}$$

Now let $L = \log(C(0))^2$, which we can (and do) take as real logarithm. As before, we now build an orthogonal $C^\infty$ function $H$ such that

$$H(t) = \begin{cases} I_{n-m_1}, & t \leq -p_1 + r_1, \\ (C(0))^2, & t \geq p_1 - r_1. \end{cases}$$

Then we let $\tilde{Q}_1(t) = \bar{Q}_1(t)H(t)$ for $-p_1 - r_1 \leq t \leq p_1 + r_1$ and have $\tilde{Q}_1$ real orthonormal and $C^l$. Moreover, it satisfies

$$\tilde{Q}_1(t) = \begin{cases} Q_1(t + p_1)(C(0))^2, & -p_1 - r_1 \leq t \leq -p_1 + r_1, \\ Q_1(t - p_1)(C(0))^2, & p_1 \leq t \leq p_1 + r_1, \end{cases}$$

so that $\tilde{Q}_1(t + 2p_1) = \tilde{Q}_1(t)$, $-p_1 \leq t \leq -p_1 + r_1$, and thus we can build a real $C^l$ orthonormal function of period $2p_1$. \hfill \square

We now give periodicity results for the SVD of a 1-periodic function allowing the singular values to coalesce. The situation is very close to what we have just proven for the Hermitian eigenproblem. We will consider the $C^k$ case of a constant rank function. The $C^\infty$ case is dealt with in a similar way (see Remark 3.26).

First, we consider the case of complex valued $A$. We have the following.

**Theorem 3.24.** Let $A \in C^k_p(\mathbb{R}, \mathbb{C}^{n \times n})$, let $\text{rank}(A) = n - r$ for all $t$, and let there exist $e \leq k$ such that for the nonzero singular values of $A$ (2.5) holds:

$$\liminf_{t \to 0} \frac{\sigma_i(t + \tau) - \sigma_j(t + \tau)}{\tau^e} \in (0, \infty)$$

for all $t$ and $i \neq j$. Then, for the matrix-valued function of singular values of $A$, $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_{n-r})$, we have $\Sigma \in C^{l}_p(\mathbb{R}, \mathbb{C}^{(n-r) \times (n-r)})$, $1 \leq p \leq \mu^{*}(n-r)$.

**Proof.** The result follows from Theorem 3.3(i) and Theorem 3.10. \hfill \square

Under the assumptions of Theorem 3.24, we know that there exist orthonormal functions $U, V$ of appropriate dimensions such that

$$U^*AV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_{n-r}).$$

Moreover, if we let $U = [U_1 \quad U_2]$ and $V = [V_1 \quad V_2]$ partitioned so that $U_1^*AV_1 = \Sigma$, then from Theorem 3.3(i) we know that $U_2 \in C^k_p(\mathbb{R}, \mathbb{C}^{m \times (m-n+r)})$, $V_2 \in C^k_p(\mathbb{R}, \mathbb{C}^{n \times r})$. We also know that $U_1 \in C^{k-r}(\mathbb{R}, \mathbb{C}^{m \times (n-r)})$, $V_1 \in C^{k-r}(\mathbb{R}, \mathbb{C}^{n \times (n-r)})$. Further, we have the following.

**Theorem 3.25.** Under the assumptions of Theorem 3.24, and with above notation, there exist orthonormal $U_1 \in C^k_p(\mathbb{R}, \mathbb{C}^{m \times (n-r)})$ and $V_1 \in C^k_p(\mathbb{R}, \mathbb{C}^{n \times (n-r)})$, so that $U = [\hat{U}_1 \quad U_2]$ and $V = [\hat{V}_1 \quad V_2]$ are unitary and $U^*AV = [\Sigma \quad 0]$.

**Proof.** Since $AV_1 = U_1\Sigma$ and $A^*U_1 = V_1\Sigma$, then

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\hat{V}_1}{\sqrt{\lambda}} \\ \frac{V_1}{\sqrt{\lambda}} \end{bmatrix} = \begin{bmatrix} \frac{\hat{V}_1}{\sqrt{\lambda}} \\ \frac{V_1}{\sqrt{\lambda}} \end{bmatrix} \Sigma.$$
Let \( Q_1 := \begin{bmatrix} \frac{1}{\sqrt{2}} U_1 & V_1 \end{bmatrix} \). As previously noticed, \( Q_1 \in C^{k-e}(\mathbb{R}, \mathbb{C}^{(m+n) \times (n-r)}) \), and \( Q_1 \) is orthonormal. So, by Theorem 3.23, we can replace \( Q_1 \) by \( \tilde{Q}_1 \in C^{k-e}_p \), still satisfying (3.16): \( \begin{bmatrix} 0 & A \end{bmatrix} \tilde{Q}_1 = \tilde{Q}_1 \Sigma \). Define \( \tilde{U}_1 \) and \( \tilde{V}_1 \), of the same dimensions as \( U_1, V_1 \), respectively, from the partition \( \tilde{Q}_1 := \begin{bmatrix} \frac{1}{\sqrt{2}} \tilde{U}_1 & \tilde{V}_1 \end{bmatrix} \). Since we have not only (3.16), but also

\[
\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} U_1 \\ \frac{1}{\sqrt{2}} V_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} U_1 \\ \frac{1}{\sqrt{2}} V_1 \end{bmatrix} (-\Sigma),
\]
then we also get that

\[
\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \tilde{U}_1 \\ \frac{1}{\sqrt{2}} \tilde{V}_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \tilde{U}_1 \\ \frac{1}{\sqrt{2}} \tilde{V}_1 \end{bmatrix} (-\Sigma).
\]

Recalling that \( \sigma_i \neq 0 \), \( i = 1, \ldots, n-r \), and arguing as in the proof of Theorem 2.4, we obtain that \( \tilde{U}_1 \) and \( \tilde{V}_1 \) are orthonormal. Using Lemma 2.2, we finally obtain that \( \tilde{U} \) and \( \tilde{V} \) are unitary. \( \square \)

**Remark 3.26.** For the SVD of \( A \in C^\omega \), we obtain much the same periodicity results as those of the \( C^k \) case, but the details differ somewhat. First, from [3] one obtains an analytic SVD of analytic \( A \). Then, we can infer periodicity of the singular values as we did for the analytic eigenvalues of a Hermitian function. Finally, we can use the construction of [9] on the analytic Hermitian matrix \( \begin{bmatrix} 0 & \tilde{A} \\ \tilde{A}^* & 0 \end{bmatrix} \), as already pointed out in Remark 3.20. The details of this construction are omitted.

To complete the periodicity results for the SVD of a matrix, we now turn to the case of a real-valued function \( A \). We have the following.

**Theorem 3.27.** Let \( A \in C^k_1(\mathbb{R}, \mathbb{R}^{m \times n}) \) and let rank(\( A \)) = \( n-r \) for all \( t \). Let \( e \leq k \) be such that for the nonzero singular values of \( A \) (3.15) holds, so that there exist orthogonal \( U \in C^{k-e}(\mathbb{R}, \mathbb{R}^{m \times m}) \) and \( V \in C^{k-e}(\mathbb{R}, \mathbb{R}^{n \times n}) \) such that \( U^*AV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \) with \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_{n-r}) \). Then we can take \( \Sigma \in C^p_2(\mathbb{R}, \mathbb{R}^{(n-r) \times (n-r)}) \), \( p \leq \mu^*(n-r) \). Moreover, we can choose \( U = \begin{bmatrix} \tilde{U}_1 & U_2 \end{bmatrix} \) and \( V = \begin{bmatrix} \tilde{V}_1 & V_2 \end{bmatrix} \), with \( AV = \begin{bmatrix} \tilde{U}_1 \Sigma & U_2 \end{bmatrix} \), and \( \tilde{U}_1 \in C^{k-e}_p(\mathbb{R}, \mathbb{R}^{m \times (n-r)}) \), \( \tilde{V}_1 \in C^{k-e}_p(\mathbb{R}, \mathbb{R}^{n \times (n-r)}) \) and \( U_2 \in C^k_2(\mathbb{R}, \mathbb{R}^{m \times (m-n+r)}) \), \( V_2 \in C^k_2(\mathbb{R}, \mathbb{R}^{n \times r}) \).

**Proof.** The main ingredient is to show the stated periodicity of the singular values’ function \( \Sigma \). To this end, we can reason as follows. The \( \sigma_i \)'s are the positive square roots of the nonzero eigenvalues of \( A^TA \): \( \sigma_i(t) = \sqrt{\lambda_i(t)} \) for all \( t \) for a given ordering of the nonzero \( C^k \) eigenvalues of \( A^TA \). In particular, the \( \sigma_i \)'s can be taken of the same period as the \( \lambda_i \)'s. Because of the assumption (3.15), and of Theorem 3.2 (in particular, (3.2) in the real, normal case), we have that there exists orthogonal function \( V = \begin{bmatrix} \tilde{V}_1 & V_2 \end{bmatrix} \) such that

\[
V^T(A^TA)V = \begin{bmatrix} \lambda_1 & \ldots & \lambda_{n-r} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 
\end{bmatrix}.
\]

Since \( A^TA \) has period 1, then reasoning as in the proof of Lemma 3.14 relatively to \( D = \text{diag}(\lambda_1, \ldots, \lambda_{n-r}) \), we obtain that \( D \) can be taken of period \( p, 1 \leq p \leq \mu^*(n-r) \).

At this point, we proceed similarly to the proof of Theorem 3.25, by using Theorem 3.3(i) and Theorem 3.23 in the real case, to obtain that \( U \) and \( V \) can be chosen as stated. \( \square \)
Acknowledgment. J. L. Chern gratefully acknowledges the hospitality received from the School of Mathematics and Center for Dynamical Systems and Nonlinear Studies at Georgia Tech for the academic year 1997–1998.

REFERENCES