(2) \[
\begin{bmatrix}
0 \\
1 \\
0 \\
\end{bmatrix}
\] neither not all the zero rows are at the bottom

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 2 \\
\end{bmatrix}
\] REF not all the pivots are 1's

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\] neither does not follow the staircase pattern

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\] REF

(3) \[
\begin{bmatrix}
1 & 1 & 3 \\
0 & 2 & 0 \\
1 & 3 & h \\
\end{bmatrix}
\] \(r_2 \Rightarrow \frac{1}{2} r_2\) \[
\begin{bmatrix}
1 & 1 & 3 \\
0 & 0 & 0 \\
1 & 3 & h \\
\end{bmatrix}
\] \(r_3 \Rightarrow r_3 - r_1\)

\[
\begin{bmatrix}
1 & 1 & 3 \\
0 & 1 & 0 \\
0 & 2 & h-3 \\
\end{bmatrix}
\] \(r_3 \Rightarrow r_3 - 2r_2\) \[
\begin{bmatrix}
1 & 1 & 3 \\
0 & 1 & 0 \\
0 & 0 & h-3 \\
\end{bmatrix}
\]

\(h = 3\) in order to make \(x_3\) a free variable. With one free variable, the solution set will be a line.
\[
\begin{bmatrix}
1 & 3 & 0 & -2 \\
0 & 0 & 1 & 4
\end{bmatrix}
\]
\begin{align*}
x_1 + 3x_2 + 0x_3 &= -2 \\
0x_1 + 0x_2 + x_3 &= 4
\end{align*}
\begin{align*}
x_1 &= -2 \\
x_2 &= -2x_2 \\
x_3 &= 4
\end{align*}
\[x_2 \text{ free (real)}\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-2 \\
-3x_2 \\
x_2 \\
4
\end{bmatrix} + x_2 \begin{bmatrix}
-3 \\
1 \\
0
\end{bmatrix}
\]

\[
A = -2 \\
B = 0 \\
C = 4
\]
\[
D = -3 \\
E = 1 \\
F = 0
\]

\[\text{A) True} \]

The solution set being a line means there is one free variable. This does not change between \(Ax=b\) and \(Ax=0\). The line is just translated to pass through 0 instead of \(b\).

\[\text{B) False} \]

Counterexample:

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]
\(Ax=b\) inconsistent

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\(Ax=0\) consistent

\[
x_1 = 0 \\
x_2 = x_2 \text{ (free)}
\]
\[
\begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & h \\
1 & 1 & h \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & h \\
1 & 1 & h \\
\end{bmatrix} \\
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
2 & 1 & h \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & h-2 \\
\end{bmatrix} \\
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & h-2 \\
\end{bmatrix}
\]

Not linear independent if there is not a pivot in every column. Thus, not linear independent if \( h = 2 \).

5x7 matrix:

\[
\begin{bmatrix}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\end{bmatrix}
\]

A) False
The maximum number of pivots would be 5.
There are 7 columns.
7 > 5, so there cannot be a pivot in every column.

B) True
The minimum number of free variables would be 2 because 7 - 5 = 2.
Thus, you would need two different vectors for each free variable in the solution set.

C) True. We can provide an example:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Thus, there is no potential for a row of zeros in \( A \).
\[ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 3 & -2 & 1 & 1 \end{bmatrix} \Rightarrow R_2 \rightarrow R_2 - 3R_1 \]

\[ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{bmatrix} \Rightarrow R_2 \rightarrow -\frac{1}{5} R_2 \]

\[ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{5} & 2 \end{bmatrix} \Rightarrow R_1 \rightarrow R_1 - R_2 \]

\[ \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \]

2 pivots and 1 free variable

Thus, the solution set is a line.
Thus, there are infinitely many solutions.

**Note:** Because there are two pivots,

\[ \text{span} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{5} \end{bmatrix} \]

would be a plane.

9. False

This is only true when the solution set contains the origin. You could have the following solution set counter-example:

\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

This would appear geometrically as:

![Graph showing a line with a counterexample vector](image)

which does not contain the origin and cannot be a span.
A) \[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 3 & 5 \\
3 & 5 & 6 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -3 & -3 \\
0 & -1 & -3 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -3 & -3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\text{TWO pivots, so Span } \mathbf{C}(A) \text{ is a NO plane in } \mathbb{R}^3
\]

B) \[
\begin{bmatrix}
1 & 8 & 2 & 0 \\
8 & 2 & 0 & 12 \\
20 & 1 & 12
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 1 & 12
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 12 \\
0 & 0 & 0
\end{bmatrix}
\text{2 pivots and 3 rows YES}
\]

C) \[
\begin{bmatrix}
1 & 1 & 0 \end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\text{2 pivots and 3 rows YES}
\]

D) This does reduce to \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
but it does not span \( \mathbb{R}^3 \) because NO there are 4 rows and thus the span is \( \mathbb{R}^4 \). The span is 3-dimensional, but it is not described as spanning \( \mathbb{R}^3 \).
3 Equations and 3 variables

Point
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
0 free variables \( \Rightarrow \) yes

Plane
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
2 free variables \( \Rightarrow \) yes

Line
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
1 free variable \( \Rightarrow \) yes

\[ \text{No solution} \]
\[ [A|b] = \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix} \]
\( \Rightarrow \) yes
**Question 12.** We can plug in each of the points (1, 2), (2, 1), and (3, 0) in the equations to see if the system is satisfied.

\[
\begin{align*}
(1, 2) & : \begin{cases} 1 + 2 = 3, \quad 2 \neq 1, \\
(2, 1) & : \begin{cases} 2 + 1 = 3, \quad 1 = 1, \\
(3, 0) & : \begin{cases} 3 + 0 = 3, \quad 0 \neq 1,
\end{cases}
\end{cases}
\end{align*}
\]

Out of the three options, only the point (2, 1) is a solution of the system.

**Question 13.** When we represent the system in matrix form \( Ax = b \), we have

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\]

Both matrices \( A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) and \( [A | b] = \begin{bmatrix} 1 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \) have the same number of pivots, so the system is consistent and has a solution. Moreover, there are two pivots, which means that there is one free variable. We conclude that the solution space is a line in \( \mathbb{R}^3 \).

Alternatively, we can consider the geometric shape of each equation of the system separately, and then the solution to the system will be the intersection of these two geometric shapes. The equation \( x + y = 2 \) is a plane in \( \mathbb{R}^3 \) parallel to the \( z \)-axis. The equation \( z = 1 \) is a plane in \( \mathbb{R}^3 \) perpendicular to the \( z \)-axis. The two planes are not parallel to each other, so they must meet at a line. So, the solution to the system is a line. The following picture is an illustration of this fact. The red line at the intersection of the two planes is the solution:

![Diagram showing the intersection of two planes representing the system of equations](image)

**Question 14.** We may plug in the given points (0, 0), (1, −1), and (0, −7) into the equation \((x^2 + y^2) + Bx + Cy + D = 0\) to get the following equations in \( B, C, \) and \( D \):

\[
\begin{cases}
(0^2 + 0^2) + 0 \cdot B + 0 \cdot C + D = 0 \\
(1^2 + (-1)^2) + 1 \cdot B + (-1) \cdot C + D = 0 \\
(0^2 + (-7)^2) + 0 \cdot B + (-7) \cdot C + D = 0
\end{cases}
\]
which simplifies to
\[
\begin{aligned}
D &= 0 \\
B - C + D &= -2 \\
-7C + D &= -49
\end{aligned}
\]
We can solve this by forming an augmented matrix and row-reducing it as
\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & -1 & 1 & -2 \\
0 & -7 & 1 & -49
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 1 & -2 \\
0 & 0 & 1 & 0 \\
0 & -7 & 1 & -49
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 1 & -2 \\
0 & 0 & 1 & 0 \\
0 & -7 & 1 & -49
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 0 & -2 \\
0 & -7 & 0 & -49 \\
0 & 0 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 0 & -2 \\
0 & 1 & 0 & 7 \\
0 & 0 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 5 \\
0 & 1 & 0 & 7 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]
so we get the solution \( B = 5, \ C = 7, \) and \( D = 0. \)

**Question 15.** We want to find a \( 3 \times 2 \) matrix \( A \) and a vector \( b \) in \( \mathbb{R}^3 \) such that the augmented matrix \( [A \mid b] \) is in RREF and the linear system \( A \begin{bmatrix} x \\ y \end{bmatrix} = b \) has the solution \( y = -3x, \) which we rewrite as \( 3x + y = 0. \)

Since there is one free variable, we see that \( A \) has exactly one pivot. Since each non-zero row of \( [A \mid b] \) has a pivot, there must be two rows of \( [A \mid b] \) which are the zero row. And these are the bottom two rows, because the matrix is in RREF.

Finally, the first row corresponds to the equation \( 3x + y = 0, \) so the first row is a multiple of \( \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}. \) For the row to have a leading 1, it must be multiplied \( 1/3 \) to get \( \begin{bmatrix} 1 & 1/3 & 0 \end{bmatrix}. \) Therefore,
\[
[A \mid b] = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

**Question 16.** False. The number of vectors says nothing about linear independence. For
example, take the set of 5 vectors
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\] in \(\mathbb{R}^7\). Clearly, this set is linearly dependent.

**Question 17.** The first two options are correct; they are indeed equivalent to stating that the columns of \(A\) are linearly independent.

The third and fourth options are not equivalent, however. It is true that if the columns of \(A\) are linearly independent, then no two columns of \(A\) are equal or multiples of each other. However, the converse does not hold; just because no two columns of \(A\) are equal or multiples of each other, that does not imply that the columns of \(A\) are linearly independent. For example, take the matrix

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

We can verify that no two columns of \(A\) are equal or multiples of each other. However, the columns of \(A\) are not linearly independent:

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} - \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(We can also verify that \(A\) does not have a pivot in the third column, and that \(Ax = 0\) has a non-zero solution \([-1, -1, 1]^T\). This is consistent with our answer that the first two options are equivalent to linear independence of the columns of \(A\).)

**Question 18.** We can form augmented matrices for each system and find the number of pivots. Then the number of free variables is the total number of variables minus the number of pivots.

The first system can be written as

\[
\begin{bmatrix}
1 & -1 & 8 & -6
\end{bmatrix}
\]

which has 1 pivot (the first column), so there are \(3 - 1 = 2\) free variables.

The second system becomes

\[
\begin{bmatrix}
1 & 3 & 0 & 0 & 0 & 7 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 5
\end{bmatrix}
\]
which has 3 pivots in columns 1, 3, and 5, so there are $5 - 3 = 2$ free variables.

The third system,
\[
\begin{bmatrix}
1 & -1 & 0 & 0 & -5 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
also has 3 pivots in columns 1, 3, and 5, so there are $6 - 3 = 3$ free variables.

**Question 19.** Note that the free variables are $x_2$ and $x_4$. We can rewrite the augmented matrix as the equations
\[
\begin{align*}
x_1 - 2x_2 - 5x_4 &= 0 \\
x_3 + x_4 &= 0
\end{align*}
\]
and we can solve for $x_1$ and $x_3$ in terms of $x_2$ and $x_4$. From the first equation, we get $x_1 = 2x_2 + 5x_4$, and from the second equation, we get $x_3 = -x_4$. Therefore we have
\[
\begin{align*}
x_1 &= 2x_2 + 5x_4 \\
x_2 &= x_2 \\
x_3 &= -x_4 \\
x_4 &= x_4
\end{align*}
\]

**Question 20.** We recognize that the solution to $Ax = b$ is a translation of the solution to $Ax = 0$. Since the plane $z = 1$ is parallel to the $xy$-plane, the solution to $Ax = 0$ is also a plane parallel to the $xy$-plane. Moreover, the solution to $Ax = 0$ includes the zero vector, so the solution passes through the origin. We conclude that the solution to $Ax = 0$ coincides with the $xy$-plane.

**Question 21.** The three nearest nodes to the node $x$ are the nodes 4, $y$, and $z$. The average of these three nodes equals $x$, so we have
\[
x = \frac{4 + y + z}{3}
\]
which simplifies to $3x - y - z = 4$. We can do the same for nodes $y$ and $z$ to get the equations
\[
y = \frac{0 + z + x}{3}, \quad z = \frac{8 + x + y}{3}
\]
which each simplifies to $x - 3y + z = 0$ and $x + y - 3z = -8$. So we get a system of 3 equations in 3 variables $x, y, z$:
\[
\begin{align*}
3x - y - z &= 4 \\
x - 3y + z &= 0 \\
x + y - 3z &= -8
\end{align*}
\]
We can now solve for \( x, y, \) and \( z \) by row-reduction:

\[
\begin{bmatrix} 3 & -1 & -1 & 4 \\ 1 & -3 & 1 & 0 \\ 1 & 1 & -3 & -8 \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 3 & -1 & -1 & 4 \\ 1 & 1 & -3 & -8 \end{bmatrix}
\]

\[
R_1 \leftrightarrow R_2 \rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 3 & -1 & -1 & 4 \\ 1 & 1 & -3 & -8 \end{bmatrix}
\]

\[
R_2 \leftrightarrow R_3 \rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 3 & -1 & -1 & 4 \\ 1 & 1 & -3 & -8 \end{bmatrix}
\]

\[
R_2 = R_2 - R_1 \\
R_3 = R_3 - 3R_1 \rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & -4 & -8 \\ 0 & 8 & -4 & 4 \end{bmatrix}
\]

\[
R_2 = (1/4)R_2 \\
R_3 = (1/4)R_3 \rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -1 & 1 \end{bmatrix}
\]

\[
R_3 = R_3 - 2R_2 \rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}
\]

\[
R_3 = R_3 - 2R_2 \rightarrow \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}
\]

\[
R_3 = R_3 - R_3 \\
R_2 = R_2 + R_3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}
\]

so \( x = 4, \ y = 3, \) and \( z = 5. \)