Math 1553 Quiz 10 Solution

Question 2

Which of the following are correct diagonalizations of the matrix \[
\begin{bmatrix}
2 & 6 \\
0 & -1
\end{bmatrix}
\]?

Ans: \[
\begin{bmatrix}
1 & -2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -2 \\
0 & 1
\end{bmatrix}^{-1}
\] and

\[
\begin{bmatrix}
-2 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & 2
\end{bmatrix}^{-1}
\] and

\[
\begin{bmatrix}
2 & 2 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
2 & 0 \\
0 & -1
\end{bmatrix}^{-1}
\]

Because the matrix is triangular, we know that the eigenvalues are \(\lambda = 2, -1\). The matrix D can therefore look like \[
\begin{bmatrix}
2 & 0 \\
0 & -1
\end{bmatrix}
\] or \[
\begin{bmatrix}
-1 & 0 \\
0 & 2
\end{bmatrix}
\] Solving for the eigenvector associated with \(\lambda = 2\), we get \(A - 2I = \begin{bmatrix}
0 & 6 \\
0 & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \rightarrow x_1 = x_1, x_2 = 0\). A basis for the 2-eigenspace is \(\begin{bmatrix}1 \\ 0\end{bmatrix}\), or any multiple of this vector (such as \(\begin{bmatrix}2 \\ 0\end{bmatrix}\)). Solving for the eigenvector associated with \(\lambda = -1\), we get \(A - (-1)I = \begin{bmatrix}
3 & 6 \\
0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix} \rightarrow x_1 = -2x_2, x_2 = x_2\). A basis for the -1-eigenspace is \(\begin{bmatrix}-2 \\ 1\end{bmatrix}\), or any multiple of this vector (such as \(\begin{bmatrix}2 \\ -1\end{bmatrix}\)). The eigenvectors in matrix C must appear in the same order (column-wise) as the respective eigenvalues in the matrix D.

Question 3

Suppose that A is a 5x5 matrix with characteristic polynomial \((1 - \lambda)^2 (3 - \lambda)^2 (\pi - \lambda)\) and also that A is diagonalizable. What is the dimension of the 1-eigenspace of A?

Ans: 2

Because the matrix A is diagonalizable, the algebraic multiplicity (number of times \(\lambda\) appears as a root in the characteristic polynomial) must match the geometric multiplicity (dimension of the eigenspace associated with \(\lambda\)) for each value of \(\lambda\). 1 appears as a root of the characteristic polynomial twice, therefore the geometric multiplicity of \(\lambda = 1\) / dimension of the 1-eigenspace is 2.
Question 4
Suppose A is a 2x2 matrix whose entries are real numbers, and suppose A has eigenvalue 1-i with corresponding eigenvector \( \begin{pmatrix} 2 \\ 1 - i \end{pmatrix} \). Which of the following must be true?

Ans: A has eigenvalue 1+i with eigenvector \( \begin{pmatrix} 2 \\ 1 + i \end{pmatrix} \)

From the slides, "if \( \lambda \) is an eigenvalue with eigenvector \( v \), then \( \bar{\lambda} \) is an eigenvalue with eigenvector \( \bar{v} \)." In this case, \( \bar{\lambda} = 1 - i = 1 + i \) and \( \bar{v} = \begin{pmatrix} 2 \\ 1 + i \end{pmatrix} \)

Question 5
If A is a diagonalizable 10 x 10 matrix, then A must have 10 distinct eigenvalues.

Ans: False

An nxn matrix could be diagonalizable without n distinct eigenvalues if at least one of the eigenvalues has a multiplicity greater than one. For instance, the the 10 x 10 identity matrix \( I_{10} \) is diagonalizable but only has one distinct eigenvalue, \( \lambda = 1 \).

Question 6
Suppose that A is a 4 x 4 matrix with eigenvalues 0, 1, and 2, where eigenvalue 2 has geometric multiplicity 2 (meaning that the dimension of the 2-eigenspace is 2). Which of the following statements much be true?

Ans: A is diagonalizable, A is not invertible

Because the geometric multiplicity of \( \lambda = 2 \) is 2, and eigenvalues 0, 1 must have at least a 1-dimensional eigenspace, we have the 4 linearly independent eigenvectors needed to complete the diagonalization of a 4x4 matrix.

Because 0 is an eigenvalue of A, the equation Ax=0 does not have only the trivial solution. Therefore, A is not invertible.