1. Let $V$ be the set of vectors in $\mathbb{R}^3$ given by \{(a,b,c) \in \mathbb{R}^3 | a= -c$ and $b=0$\}.

(a). Does $V$ contain the 0 vector?  
\[ (0,0,0) \] satisfies that $a=-c$, $b=0$, and $b=0$

(b). Is $V$ closed under addition?  
Yes.

Soy $U: (a_1, b_1, c_1)$ and $V: (a_2, b_2, c_2)$ are in $V$.

Then, we must have $b_1=b_2=0$, $a_1=-c_1$, $a_2=-c_2$

Then, as $a_1+a_2=-c_1-c_2$, $b_1+b_2=0$, so $U+V: (a_1+a_2, b_1+b_2, c_1+c_2)$ is still in $V$.

(c). Is $V$ closed under multiplication?  
Yes.

Soy $U: (a_1, b_1, c_1)$ is in $V$.

Then, we must have $a_1=-c_1$, $b_1=0$.

Then, if we multiply a real number $k$ to $U$: $kU = (ka_1, kb_1, kc_1)$, where $k,b,=0$, and $ka_1=-kc_1$.

So, $kU$ is still in $V$.

(d). Is $V$ a subspace of $\mathbb{R}^3$?  
Yes.

Based on definition:  
1. Zero-vector is in $V$ - meet  
2. Closure under addition - meet  
3. Closure under scalar multiplication - meet

So, $V$ is a subspace of $\mathbb{R}^3$. 
2. Let $V$ be the set of vectors in $\mathbb{R}^3$ given by $\{ (a, b, c) \in \mathbb{R}^3 \mid c \geq 0 \}$

(a). Does $V$ contain the 0 vector?  
\[ (0,0,0) \text{ satisfies that } c \geq 0 \]

(b). Is $V$ closed under addition?  
Yes

Say $U: (a_1, b_1, c_1)$ and $V: (a_2, b_2, c_2)$ are in $V$.
Then, we must have $a_1 \geq 0$, $c_1 \geq 0$
Then, as $a_1 + a_2 \geq 0$, so $U + V: (a_1 + a_2, b_1 + b_2, c_1 + c_2)$ is still in $V$.

(c). Is $V$ closed under multiplication?  
No

Say $U: (a_1, b_1, c_1)$ is in $V$.
Then, we must have $c_1 \geq 0$.
Then, if we multiply a real number $k$ to $U$: $kU = (ka_1, kb_1, kc_1)$,
where $kc_1$ might be negative (i.e. $k = -1$), in which case $kU$ is not in $V$.

(d). Is $V$ a subspace of $\mathbb{R}^3$?  
No

Based on definition:  
1. Zero-vector is in $V$  
   - meet
2. Closure under addition  
   - meet
3. Closure under scalar multiplication  
   - does not meet

So, $V$ is not a subspace of $\mathbb{R}^3$. 
3. When is the set of solutions to a matrix equation a subspace?

Example:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

In this case, \(x = 0\), \(y\) and \(z\) are free variables.

The set of solutions are:

\[
\begin{bmatrix}
y \\
1 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}z
\]

So, \(\text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) meets the requirement of:

1. \((\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})\) is in the subspace
2. closed under addition
3. closed under scalar multiplication.

So, \(\text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) is a subspace in \(\mathbb{R}^3\).

Counter-Example:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

In this case, \(x = 1\), \(y\) and \(z\) are free variables.

The set of solutions are:

\[
\begin{bmatrix}
y \\
1 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}z + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

The set of solutions is not a subspace because \((\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})\) is not in the set.

Combining both examples, we have the choice "sometimes."
Observe that in the row-reduced matrix:
\[
\begin{pmatrix}
1 & 0 & 0 & -\frac{1}{3} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \frac{1}{3} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
"Red" representing pivots.

So, first, second, third columns are pivotal columns.

Since, the pivot columns of a matrix \(A\) form a basis for \(\text{Col}(A)\).

As we go back to the previous matrix, the first, second, third columns are the vectors spanning \(\text{Col}(A)\).

So, the solution should be:
\[
\begin{bmatrix}
(1) \\
(5) \\
(1)
\end{bmatrix}
+ \begin{bmatrix}
(2) \\
(4) \\
(2)
\end{bmatrix}
\begin{bmatrix}
(1) \\
(6) \\
(2)
\end{bmatrix}
\]

Note: \(\begin{bmatrix}
(0) \\
(0) \\
(0)
\end{bmatrix}\) does not span the \(\text{Col}(A)\), because \(\text{Col}(A)\) contains vectors whose last coordinate is nonzero.
5. A’s row reduced echelon form is as follows: 
\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The free variables are \(x_2\) and \(x_4\). The parametric form of the solution set is:

\[
\begin{align*}
    x_1 &= -x_4 \\
    x_2 &= x_4 + 1 \\
    x_3 &= -2x_4 \\
    x_4 &= x_4
\end{align*}
\]

Therefore, \(\text{Null}(A) = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \)