Reduced Row Echelon Form

Poll

Which are in reduced row echelon form?

\begin{align*}
\begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix} & \quad \text{not 1} \\
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix} & \quad \text{not even REF} \\
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 8 & 0
\end{pmatrix} & \quad \text{YES} \\
\begin{pmatrix}
1 & 17 & 0 \\
0 & 0 & 1
\end{pmatrix} & \quad \text{YES} \\
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix} & \quad \text{YES}
\end{align*}

REF:
1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

RREF:
4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column
Announcements Aug 26

- WeBWorK on Section 1.1 due Thursday night
- Quiz on Section 1.1 Friday 8 am - 8 pm EDT
- My office hours Tue 11-12, Thu 2-3, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Juntao Thu 3-4

- Studio on Friday
- Stay tuned for PLUS session info
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- For general questions, post on Teams/Piazza
- Find a group to work with - let me know if you need help
Section 1.2

Row reduction
Row Reduction and Echelon Forms

A matrix is in **row echelon form** if

1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

\[
\begin{pmatrix}
\star & * & * & * & * \\
0 & \star & * & * & * \\
0 & 0 & 0 & \star & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

This system is easy to solve using back substitution.

The **pivot** positions are the leading entries in each row.
Reduced Row Echelon Form

A system is in **reduced row echelon form** if also:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

For example:

\[
\begin{pmatrix}
1 & 0 & \ast & 0 & \ast \\
0 & 1 & \ast & 0 & \ast \\
0 & 0 & 0 & 1 & \ast \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

This system is even easier to solve.

**Important.** In any discussion of row echelon form, we ignore any vertical lines!

Can every matrix be put in reduced row echelon form?
Row Reduction Algorithm

To find row echelon form:

Step 1  Swap rows so a leftmost nonzero entry is in 1st row (if needed)
Step 2  Scale 1st row so that its leading entry is equal to 1
Step 3  Use row replacement so all entries below this 1 (or, pivot) are 0

Then cover the first row and repeat the three steps.

To then find reduced row echelon form:

- Use row replacement so that all entries above the pivots are 0.

Examples.

\[
\begin{pmatrix}
1 & 2 & 3 & 9 \\
2 & -1 & 1 & 8 \\
3 & 0 & -1 & 3
\end{pmatrix}
\quad
\begin{pmatrix}
0 & 7 & -4 & 2 \\
2 & 4 & 6 & 12 \\
3 & 1 & -1 & -2
\end{pmatrix}
\quad
\begin{pmatrix}
4 & -5 & 3 & 2 \\
1 & -1 & -2 & -6 \\
4 & -4 & -14 & 18
\end{pmatrix}
\]
Solutions of Linear Systems

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

\[
\begin{pmatrix}
1 & 0 & 5 \\
0 & 1 & 2
\end{pmatrix}
\]

What are the solutions? Say the variables are \( x \) and \( y \).

\[ x = 5 \]
\[ y = 2 \]

\((5, 2)\)
Solutions of Linear Systems: Consistency

Solve the linear system associated to:

\[
\begin{align*}
\text{1st eqn} & : \begin{pmatrix} 1 & 0 & 5 & | & 0 \end{pmatrix} \\
\text{2nd eqn} & : \begin{pmatrix} 0 & 0 & 0 & | & 1 \end{pmatrix}
\end{align*}
\]

Say the variables are \(x, y,\) and \(z\).

\[x + 5z = 0\]
\[0 = 1\]

Inconsistent! = no soln.

A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

gives: \(0 = 1\)
Example with a parameter

For which values of $h$ does the following system have a solution?

\[
\begin{align*}
  x + y &= 1 \\
  2x + 2y &= h
\end{align*}
\]

Solve this by row reduction and also solve it by thinking geometrically.

Want no pivot in last col. 
So $h - 2 = 0$ 
So $h = 2$
1.3 Parametric Form
Outline of Section 1.3

- Find the parametric form for the solutions to a system of linear equations.
- Describe the geometric picture of the set of solutions.
Free Variables

We know how to understand the solution to a system of linear equations when every column to the left of the vertical line has a pivot. For instance:

\[
\begin{pmatrix}
1 & 0 & | & 5 \\
0 & 1 & | & 2
\end{pmatrix}
\]

If the variables are \(x\) and \(y\) what are the solutions?

(5, 2)
Free Variables

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? For instance:

\[
\begin{pmatrix}
1 & 0 & 5 & 0 \\
0 & 1 & 2 & 1 \\
\end{pmatrix}
\]

represents two equations:

\[
\begin{align*}
x_1 + 5x_3 &= 0 \\
x_2 + 2x_3 &= 1 \\
\end{align*}
\]

There is one free variable \(x_3\), corresponding to the non-pivot column. To solve, we move the free variable to the right:

\[
\begin{align*}
x_1 &= -5x_3 \\
x_2 &= 1 - 2x_3 \\
x_3 &= x_3 \text{ (free; any real number)} \\
\end{align*}
\]

This is the parametric solution. We can also write the solution as:

\((-5x_3, 1 - 2x_3, x_3)\)

What is one particular solution? What does the set of solutions look like?
Free Variables

Solve the system of linear equations in \(x_1, x_2, x_3, x_4\):

\[
\begin{align*}
  x_1 + 5x_3 &= 0 \\
  x_4 &= 0
\end{align*}
\]

So the associated matrix is:

\[
\begin{pmatrix}
  1 & 0 & 5 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

To solve, we move the free variable to the right:

\[
\begin{align*}
  x_1 &= -5x_3 \\
  x_2 &= x_2 \quad \text{(free)} \\
  x_3 &= x_3 \quad \text{(free)} \\
  x_4 &= 0
\end{align*}
\]

Or: \((-5x_3, x_2, x_3, 0)\). This is a plane in \(\mathbb{R}^4\).

The original equations are the implicit equations for the solution. The answer to this question is the parametric solution.
If we have a consistent system of linear equations, with \( n \) variables and \( k \) free variables, then the set of solutions is a \( k \)-dimensional plane in \( \mathbb{R}^n \).

Why does this make sense?
A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. two points
4. line
5. plane
6. 3-dimensional plane
7. 4-dimensional plane
 Implicit versus parametric equations of planes

Find a parametric description of the plane

\[ x + y + z = 1 \]

Step 1
Make a matrix
\[
\begin{pmatrix}
1 & 1 & 1 & 1
\end{pmatrix}
\]

Step 2
RREF
\[
\begin{pmatrix}
1 & 1 & 1 & 1
\end{pmatrix}
\]

Step 3
Find free vars, etc.
\[ x = 1 - y - z \]
\[ y = y \text{ (free)} \]
\[ z = z \text{ (free)} \]

\[ (1 - y - z, y, z) \]

The original version is the implicit equation for the plane. The answer to this problem is the parametric description.

Also correct:
\[ (x, 1 - x - z, z) \]

But doesn't follow our recipe.
Typical exam questions

True/False: If a system of equations has 100 variables and 70 equations, then there must be infinitely many solutions.

False. Maybe inconsistent.

True/False: If a system of equations has 70 variables and 100 equations, then it must be inconsistent.

False. Maybe all eqns same (or multiples).

How can we tell if an augmented matrix corresponds to a consistent system of linear equations?

No pivot in last column.

If a system of linear equations has finitely many solutions, what are the possible numbers of solutions?

0 or 1
Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of system of linear equations.

1. **The last column is a pivot column.**
   \[ \begin{pmatrix}
   1 & 0 & \vdots \\
   0 & 1 & \vdots \\
   0 & 0 & 1
   \end{pmatrix} \]
   \[ \text{thus the system is } \text{inconsistent}. \]

2. **Every column except the last column is a pivot column.**
   \[ \begin{pmatrix}
   1 & 0 & 0 & \star \\
   0 & 1 & 0 & \star \\
   0 & 0 & 1 & \star 
   \end{pmatrix} \]
   \[ \text{thus the system has a } \text{unique solution}. \]

3. **The last column is not a pivot column, and some other column isn't either.**
   \[ \begin{pmatrix}
   1 & \star & 0 & \star & \star \\
   0 & 0 & 1 & \star & \star 
   \end{pmatrix} \]
   \[ \text{thus the system has } \text{infinitely many solutions}; \text{ free variables correspond to columns without pivots}. \]
RREF
4x4, not a pivot
aug mat, in each col,
inconsistent

Want pivots in cols 1, 2, 4

\[
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[ c_1 = 1 \]

need to be 0.
incons.
cons.