Announcements Nov 11

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on 5.4, 5.5 due Thursday night
- Quiz on Sections 5.4, 5.5 Friday 8 am - 8 pm EDT
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- Writing assignment due Nov 24 (make sure to read the emails I sent)
- Office hours this week Tuesday 4-5, Thursday 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4

- Studio on Friday
- Tutoring: http://tutoring.gatech.edu/tutoring
- PLUS sessions: http://tutoring.gatech.edu/plus-sessions
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- Counseling center: https://counseling.gatech.edu

Note: Part of 5.5 got cut
Dynamics
Block diag.
Chapter 6
Orthogonality
Section 6.1
Dot products and Orthogonality
Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can’t solve $Ax = b$? How can we solve it as closely as possible?

The answer relies on orthogonality.

$\hat{b}$ is closest pt to $b$ in $\text{Col}(A)$

Can’t solve $Ax = b$. Solve $Ax = \hat{b}$ instead.
Outline

- Dot products
- Length and distance
- Orthogonality
Dot product

Say \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \) are vectors in \( \mathbb{R}^n \)

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{v} &= \sum_{i=1}^{n} u_i v_i \\
&= u_1v_1 + \cdots + u_n v_n \\
&= \mathbf{u}^T \mathbf{v}
\end{align*}
\]

*Example.* Find \( (1, 2, 3) \cdot (4, 5, 6) \).

\[
= (1 \quad 2 \quad 3) \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}
\]

\[
= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6
\]
Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \iff u = 0$

$u \cdot u = u_1^2 + u_2^2 + \ldots + u_n^2$
Length

Let \( v \) be a vector in \( \mathbb{R}^n \)

\[
\|v\| = \sqrt{v \cdot v} \quad \text{and} \quad \|v\|^2 = v \cdot v.
\]

\( = \) length of \( v \)

Why? Pythagorean Theorem

Fact. \( \|cv\| = |c| \cdot \|v\| \)

\( v \) is a unit vector of \( \|v\| = 1 \)

Problem. Find the unit vector in the direction of \((1, 2, 3, 4)\).

\[
\text{answer:} \quad \frac{V}{\|V\|} = \frac{1}{\sqrt{1^2+2^2+3^2+4^2}} \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) = \frac{1}{\sqrt{30}} \left( \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \right) = \left( \begin{array}{c} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{3}{\sqrt{30}} \end{array} \right)
\]
Distance

The distance between $v$ and $w$ is the length of $v - w$ (or $w - v$).

Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$.

\[
\| (1, 1, 1) - (1, 4, -3) \| = \| (0, 3, 4) \| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5
\]
Orthogonality

Fact. \( u \perp v \iff u \cdot v = 0 \)

Why? Pythagorean theorem again!

\[
\begin{align*}
  u \perp v & \iff \|u\|^2 + \|v\|^2 = \|u - v\|^2 \\
  & \iff u\cdot u + v\cdot v = u\cdot (u - 2u \cdot v) + v\cdot v \\
  & \iff u \cdot v = 0
\end{align*}
\]

Problem. Find a vector in \( \mathbb{R}^3 \) orthogonal to \((1, 2, 3)\).

\[
\begin{pmatrix}
  -1 \\
  3 \\
  2 \\
\end{pmatrix} \perp \begin{pmatrix}
  1 \\
  2 \\
  3 \\
\end{pmatrix}
\]

Since
\[
\begin{pmatrix}
  -13 \\
  2 \\
  3 \\
\end{pmatrix} \cdot \begin{pmatrix}
  1 \\
  2 \\
  3 \\
\end{pmatrix} =
\]

\[
-13 + 4 + 9 = 0.
\]
Summary of Section 6.1

- \( u \cdot v = \sum u_i v_i \)
- \( u \cdot u = \|u\|^2 \) (length of \( u \) squared)
- The unit vector in the direction of \( v \) is \( v/\|v\| \).
- The distance from \( u \) to \( v \) is \( \|u - v\| \)
- \( u \cdot v = 0 \iff u \perp v \)
Section 6.2
Orthogonal complements
Outline of Section 6.2

- Orthogonal complements
- Computing orthogonal complements
Orthogonal complements

$W = \text{subspace of } \mathbb{R}^n = \text{plane thru } 0$.

$W^\perp = \{v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W\}$

**Question.** What is the orthogonal complement of a line in $\mathbb{R}^3$?
Orthogonal complements

$W = \text{subspace of } \mathbb{R}^n$

$W^\perp = \{v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \text{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \in \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of $W$ and $W^\perp$ is $\{0\}$. 

$\begin{align*}
    u \cdot w = 0 & \quad \implies \quad cu \cdot w = 0 \\
    u \cdot w = 0 & \quad \& \quad v \cdot w = 0 \quad \implies \quad (u+v) \cdot w = 0.
\end{align*}$
Orthogonal complements
Finding them

Problem. Let $W = \text{Span}\{(1, 1, -1)\}$. Find the equation of the plane $W^\perp$.

What equation(s) are we solving?

$(x, y, z)$ in $W^\perp$ means: $(1, 1, -1) \cdot (x, y, z) = 0$.

$x + y - z = 0$.

Find a basis for $W^\perp$.

$x = -y + 2z$
$y = y$
$z = z$

$y \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\{\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$
Orthogonal complements

Finding them

Problem. Let $W = \text{Span}\{(1, 1, -1), (-1, 2, 1)\}$. Find a system of equations describing the line $W^\perp$.

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } W^\perp \text{ means: } \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0
\]

Find a basis for $W^\perp$.

\[
\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{cases} x = \frac{7}{2} \\ y = 0 \\ z = 7 \end{cases}
\]
Orthogonal complements
Finding them

**Recipe.** To find (basis for) $W^\perp$, find a basis for $W$, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \iff x$ is orthogonal to each row of $A$

\[
\begin{pmatrix}
1 & 1 & -1 \\
-1 & 2 & 1
\end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}
\]
Orthogonal complements
Finding them

Problem. Let $W = \text{Span}\{(1, 1, -1), (-1, 2, 1)\}$. Find a system of equations describing the line $W^\perp$.

Why? $Ax = 0 \iff x$ is orthogonal to each row of $A$

Theorem. $A = m \times n$ matrix

\[(\text{Row } A)^\perp = \text{Nul } A\]

Geometry $\leftrightarrow$ Algebra

(The row space of $A$ is the span of the rows of $A$.)
Orthogonal decomposition

Fact. Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $v$ in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where $v_W$ is in $W$ and $v_{W^\perp}$ is in $W^\perp$.

Why? Say that $w_1 + w'_1 = w_2 + w'_2$ where $w_1$ and $w_2$ are in $W$ and $w'_1$ and $w'_2$ are in $W^\perp$. Then $w_1 - w_2 = w'_2 - w'_1$. But the former is in $W$ and the latter is in $W^\perp$, so they must both be equal to 0.

Next time: Find $v_W$ and $v_{W^\perp}$. 
Orthogonal Projections

Many applications, including:
Summary of Section 6.2

- \( W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \} \)

- **Facts:**
  1. \( W^\perp \) is a subspace of \( \mathbb{R}^n \)
  2. \((W^\perp)^\perp = W\)
  3. \( \dim W + \dim W^\perp = n \)
  4. If \( W = \text{Span}\{w_1, \ldots, w_k\} \) then
     \( W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w_i \text{ for all } i \} \)
  5. The intersection of \( W \) and \( W^\perp \) is \( \{0\} \).

- To find \( W^\perp \), find a basis for \( W \), make those vectors the rows of a matrix, and find the null space.

- Every vector \( v \) can be written uniquely as \( v = v_W + v_{W^\perp} \) with \( v_W \) in \( W \) and \( v_{W^\perp} \) in \( W^\perp \)
Typical Exam Questions 6.2

- What is the dimension of $W$ if $W$ is a line in $\mathbb{R}^{10}$?
- What is $W$ if $W$ is the line $y = mx$ in $\mathbb{R}^2$?
- If $W$ is the $x$-axis in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write $v$ as $v_W + v_{W^\perp}$.
- If $W$ is the line $y = x$ in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write $v$ as $v_W + v_{W^\perp}$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ in $\mathbb{R}^4$.
- What is the orthogonal complement of $x_1x_2$-plane in $\mathbb{R}^4$?