Announcements Nov 16

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on 5.6, 6.1 due Thursday night
- No more quizzes
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- Writing assignment due Nov 24 (make sure to read the emails I sent)
- Office hours Tuesday 11-12, Thursday 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- No Studio on Friday
- Tutoring: http://tutoring.gatech.edu/tutoring
- PLUS sessions: http://tutoring.gatech.edu/plus-sessions
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- Counseling center: https://counseling.gatech.edu
Chapter 6
Orthogonality
Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can’t solve $Ax = b$? How can we solve it as closely as possible?

The answer relies on orthogonality.

Solve $Ax = \hat{b}$ instead.
Section 6.2
Orthogonal complements
Orthogonal complements

\[ W = \text{subspace of } \mathbb{R}^n = \text{plane thru } 0. \]
\[ W^{\perp} = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \} \]

**Question.** What is the orthogonal complement of a line in \( \mathbb{R}^3 \)?

What about the orthogonal complement of a plane in \( \mathbb{R}^3 \)?
Orthogonal complements

\( W = \text{subspace of } \mathbb{R}^n \)
\( W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \} \)

Facts.

1. \( W^\perp \) is a subspace of \( \mathbb{R}^n \) (it’s a null space!)
2. \( (W^\perp)^\perp = W \)
3. \( \dim W + \dim W^\perp = n \) (rank-nullity theorem!)
4. If \( W = \text{Span}\{w_1, \ldots, w_k\} \) then
   \( W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w_i \text{ for all } i \} \)
5. The intersection of \( W \) and \( W^\perp \) is \( \{0\} \).

For items 1 and 3, which linear transformation do we use?

Orthog. proj to \( W \) \quad range: \( W \) 
null space: \( W^\perp \)
Orthogonal complements
Finding them

**Recipe.** To find (basis for) $W^\perp$, find a basis for $W$, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \iff x$ is orthogonal to each row of $A$

In other words:

**Theorem.** $A = m \times n$ matrix

\[
\begin{align*}
\alpha \quad (\text{Col } A^T)^\perp & = \text{Nul } A \\
\lor \quad (\text{Col } A^T) & = (\text{Nul } A)^\perp
\end{align*}
\]

Geometry $\leftrightarrow$ Algebra

(The row space of $A$ is the span of the rows of $A$.)

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

Example:

$W = \text{Span}\{ (1, \frac{1}{3}) \}$

$W = \text{Row } (A)$

$A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3
\end{pmatrix}$

$W^\perp = (\text{Row } A)^\perp = \text{Nul } (A)$

Why?
Orthogonal decomposition

**Fact.** Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $v$ in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where $v_W$ is in $W$ and $v_{W^\perp}$ is in $W^\perp$.

**Why?**

Next time: Find $v_W$ and $v_{W^\perp}$.

$V_w$ is... orthog. proj to $W$

Will give a formula.
Orthogonal Projections

Many applications, including:
Section 6.3
Orthogonal projection
Outline of Section 6.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- Matrices for projections
- Properties of projections
Orthogonal Projections

Let $b$ be a vector in $\mathbb{R}^n$ and $W$ a subspace of $\mathbb{R}^n$.

The orthogonal projection of $b$ onto $W$ the vector obtained by drawing a line segment from $b$ to $W$ that is perpendicular to $W$.

**Fact.** The following three things are all the same:
- The orthogonal projection of $b$ onto $W$
- The vector $b_W$ (the $W$-part of $b$) algebra!
- The closest vector in $W$ to $b$ geometry!
Orthogonal Projections

Theorem. Let $W = \text{Col}(A)$. For any vector $b$ in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection $b_W$ is equal to $A x$ where $x$ is any solution.

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Step 1. Find $A^T A$ & $A^T b$

Step 2. Solve $(A^T A)x = A^T b$

(This is an $Ax = b$ problem)

Step 3. $A \cdot (\text{any solution}) = b_W$

\[T \text{ means transpose:}\]

\[
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}
\]
Orthogonal Projections

Theorem. Let $W = \text{Col}(A)$. For any vector $b$ in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T b$$

is consistent and the orthogonal projection $b_W$ is equal to $Ax$ where $x$ is any solution.

Why? Choose $\hat{x}$ so that $A\hat{x} = b_W$. We know $b - b_W = b - A\hat{x}$ is in $W^\perp = \text{Nul}(A^T)$ and so

$$0 = A^T (b - A\hat{x}) = A^T b - A^T A\hat{x}$$

$$\sim A^T A\hat{x} = A^T b$$

Note that $\text{Nul} A = (\text{Row} A)^\perp$ and $\text{Nul} A^T = (\text{Row} A^T)^\perp = (\text{Col} A)^\perp$. 
Orthogonal Projections

**Theorem.** Let $W = \text{Col}(A)$. For any vector $b$ in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection $b_W$ is equal to $A x$ where $x$ is any solution.

What does the theorem give when $W = \text{Span}\{u\}$ is a line?

$W = \text{Span}\{\left(\frac{1}{3}\right)\}$

$b = \left(\frac{4}{5}\right)$

$A = \left(\frac{1}{3}\right) = u$

**Step 1.** $A^T A = u \cdot u = ||u||^2$

$A^T b = u \cdot b$

**Step 2.** Solve $A^T A x = A^T b$

$||u||^2 \cdot x = u \cdot b$

$x = \frac{u \cdot b}{||u||^2}$

**Step 3.** Multiply

$\frac{u \cdot b}{||u||^2} u$

**Scalar mult.**
Orthogonal Projection onto a line

Special case. Let $L = \text{Span}\{u\}$. For any vector $b$ in $\mathbb{R}^n$ we have:

$$b_L = \frac{u \cdot b}{u \cdot u} u$$

Find $b_L$ and $b_{L\perp}$ if $b = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

$$\frac{u \cdot b}{u \cdot u} u = \frac{-2}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = b_L$$

$$b_{L\perp} = b - b_L = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{7}{3} \\ -\frac{5}{3} \end{pmatrix}$$
Orthogonal Projections

**Theorem.** Let $W = \text{Col}(A)$. For any vector $b$ in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection $b_W$ is equal to $A x$ where $x$ is any solution.

**Example.** Find $b_W$ if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

**Steps.** Find $A^T A$ and $A^T b$, then solve for $x$, then compute $A x$.

**Question.** How far is $b$ from $W$?
Orthogonal Projections

Example. Find $b_W$ if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T b$, then solve for $x$, then compute $Ax$.

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Step 1. $A^T A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

$A^T b = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{6}{5} \\ \frac{5}{4} \\ \frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$

Step 2. Solve $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} x = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \Rightarrow x = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$

Question. How far is $b$ from $W$?

Step 3. $(\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}) \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{7}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$

$\|b_W\| = \left\| b - b_W \right\| = \left\| \begin{pmatrix} \frac{6}{5} \\ \frac{5}{4} \\ \frac{4}{3} \end{pmatrix} - \begin{pmatrix} \frac{7}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \right\| = \left\| \begin{pmatrix} \frac{3}{20} \\ \frac{4}{4} \\ \frac{1}{3} \end{pmatrix} \right\| = \sqrt{\frac{3}{4}}$
Orthogonal Projections

**Theorem.** Let $W = \text{Col}(A)$. For any vector $b$ in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T b$$

is consistent and the orthogonal projection $b_W$ is equal to $Ax$ where $x$ is any solution.

**Special case.** If the columns of $A$ are independent then $A^T A$ is invertible, and so

$$b_W = A(A^T A)^{-1} A^T b.$$  

Why? The $x$ we find tells us which linear combination of the columns of $A$ gives us $b_W$. If the columns of $A$ are independent, there’s only one linear combination.
Let $W$ be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Then

- $T$ is a linear transformation
- $T(b) = b$ if and only if $b$ is in $W$
- $T(b) = 0$ if and only if $b$ is in $W^\perp$
- $T \circ T = T$
- The range of $T$ is $W$
Matrices for projections

**Fact.** If the columns of $A$ are independent and $W = \text{Col}(A)$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is orthogonal projection onto $W$ then the standard matrix for $T$ is:

$$A(A^T A)^{-1} A^T.$$  

**Why?** Two slides ago we said

$$A(A^T A)^{-1} A^T b = b_w$$

**Example.** Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$
Properties of projection matrices

Let \( W \) be a subspace of \( \mathbb{R}^n \) and let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be the function given by \( T(b) = b_W \) (orthogonal projection). Let \( P \) be the standard matrix for \( T \). Then

- The 1–eigenspace of \( P \) is \( W \) (unless \( W = 0 \))
- The 0–eigenspace of \( P \) is \( W^\perp \) (unless \( W = \mathbb{R}^n \))
- \( P^2 = A \)
- \( \text{Col}(P) = W \)
- \( \text{Nul}(P) = W^\perp \)
- \( A \) is diagonalizable; its diagonal matrix has \( m \) 1’s & \( n - m \) 0’s
  where \( m = \text{dim} \ W \)

You can check these properties for the matrix in the last example. It would be very hard to prove these facts without any theory. But they are all easy once you know about linear transformations!
Summary of Section 6.3

- The **orthogonal projection** of $b$ onto $W$ is $b_W$
- $b_W$ is the closest point in $W$ to $b$.
- The distance from $b$ to $W$ is $\|b_W\|$.
- **Theorem.** Let $W = \text{Col}(A)$. For any $b$, the equation $A^T Ax = A^T b$ is consistent and $b_W$ is equal to $Ax$ where $x$ is any solution.
- **Special case.** If $L = \text{Span}\{u\}$ then $b_L = \frac{u \cdot b}{u \cdot u} u$
- **Special case.** If the columns of $A$ are independent then $A^T A$ is invertible, and so $b_W = A(A^T A)^{-1} A^T b$
- When the columns of $A$ are independent, the standard matrix for orthogonal projection to $\text{Col}(A)$ is $A(A^T A)^{-1} A^T$
- Let $W$ be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$. Then
  - $T$ is a linear transformation
  - etc.
- If $P$ is the standard matrix then
  - The 1–eigenspace of $P$ is $W$ (unless $W = 0$)
  - etc.
Typical Exam Questions 6.3

• True/false. The solution to $A^T A x = A^T b$ is the point in $\text{Col}(A)$ that is closest to $b$.

• True/false. If $v$ and $w$ are both solutions to $A^T A x = A^T b$ then $v - w$ is in the null space of $A$.

• Find $b_L$ and $b_{L\perp}$ if $b = (1, 2, 3)$ and $L$ is the span of $(1, 2, 1)$.

• Find $b_W$ if $b = (1, 2, 3)$ and $W$ is the span of $(1, 2, 1)$ and $(1, 0, 1)$. Find the distance from $b$ to $W$.

• Find the matrix $A$ for orthogonal projection to the span of $(1, 2, 1)$ and $(1, 0, 1)$. What are the eigenvalues of $A$? What is $A^{100}$?