Announcements Sep 2

- WeBWorK on Sections 1.2 and 1.3 due Thursday night
- Quiz on Sections 1.2 and 1.3 Friday 8 am - 8 pm EDT
- First Midterm Sep 18
- My office hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Juntao Thu 3-4
- Studio on Friday
- Stay tuned for PLUS session info
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: https://counseling.gatech.edu
Chapter 2

System of Linear Equations: Geometry
Where are we?

In Chapter 1 we learned how to completely solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution.

In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning.
Section 2.2

Vector Equations and Spans
Span

Essential vocabulary word!

\[
\text{Span}\{v_1, v_2, \ldots, v_k\} = \left\{x_1 v_1 + x_2 v_2 + \cdots + x_k v_k \mid x_i \text{ in } \mathbb{R}\right\}
\]

= the set of all linear combinations of vectors \(v_1, v_2, \ldots, v_k\)

= plane through the origin and \(v_1, v_2, \ldots, v_k\).

Four ways of saying the same thing:

- \(b\) is in \(\text{Span}\{v_1, v_2, \ldots, v_k\}\) ← geometry
- \(b\) is a linear combination of \(v_1, \ldots, v_k\)
- the vector equation \(x_1 v_1 + \cdots + x_k v_k = b\) has a solution ← algebra
- the system of linear equations corresponding to

\[
\begin{pmatrix}
| & | & | & | & | \\
v_1 & v_2 & \cdots & v_k & b \\
| & | & | & | & |
\end{pmatrix},
\]

is consistent.
Application: Additive Color Theory

Consider now the two colors

\[
\begin{pmatrix}
180 \\
50 \\
200
\end{pmatrix}, \quad \begin{pmatrix}
100 \\
150 \\
100
\end{pmatrix}
\]

For which \( h \) is \((116, 130, h)\) in the span of those two colors?

\[
\begin{pmatrix}
180 & 100 & 116 \\
50 & 150 & 130 \\
200 & 100 & h
\end{pmatrix}
\]

row reduce

\[
\begin{pmatrix}
* & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

need a zero.

Polls channel
Consider now the two colors

\[
\begin{pmatrix}
180 \\ 50 \\ 200
\end{pmatrix}, \quad
\begin{pmatrix}
100 \\ 150 \\ 100
\end{pmatrix}
\]

For which \( h \) is \((116, 130, h)\) in the span of those two colors?

\[
\begin{pmatrix}
180 \\ 50 \\ 200
\end{pmatrix} \mapsto \begin{pmatrix}
5 \\ 15 \\ 0
\end{pmatrix}
\begin{pmatrix}
100 \\ 150 \\ 100 \\
116 \\ 130 \\ 13
\end{pmatrix} \mapsto \begin{pmatrix}
5 \\ 15 \\ 13
\end{pmatrix}
\begin{pmatrix}
0 \\ -440 \\ -352
\end{pmatrix} \mapsto \begin{pmatrix}
0 \\ -500 \\ h-520
\end{pmatrix}
\]

Bottom right: \( h - 520 - 352 \left(-\frac{500}{440}\right) = 0 \)

\( \Rightarrow h = 120 \)
Section 2.3
Matrix equations
Outline Section 2.3

- Understand the equivalences:
  
  linear system $\iff$ augmented matrix $\iff$ vector equation $\iff$ matrix equation

- Understand the equivalence:

  \[ Ax = b \] is consistent $\iff$ \[ b \] is in the span of the columns of \[ A \]

  (also: what does this mean geometrically)

- Learn for which \[ A \] the equation \[ Ax = b \] is always consistent

- Learn to multiply a vector by a matrix
Multiplying Matrices

matrix × column: \[
\begin{pmatrix}
x_1 & x_2 & \cdots & x_n
\end{pmatrix}
\begin{pmatrix}
b_1 \\ \vdots \\ b_n
\end{pmatrix} =
\begin{pmatrix}
b_1 x_1 & b_2 x_2 & \cdots & b_n x_n
\end{pmatrix}
\]

Read this as: \( b_1 \) times the first column \( x_1 \) is the first column of the answer, \( b_2 \) times \( x_2 \) is the second column of the answer...

Example:

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
7 \\
8
\end{pmatrix} =
\begin{pmatrix}
7 \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix} + 8 \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix} \\
\end{pmatrix} =
\begin{pmatrix}
7 & 16 \\
21 & 48
\end{pmatrix} =
\begin{pmatrix}
23 \\
35
\end{pmatrix}
\]
Multiplying Matrices

Another way to multiply

row vector $\times$ column vector: \[
\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\
\vdots \\
b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n
\]

Example:

\[
\begin{pmatrix} 1 & 2 \\
3 & 4 \\
5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\
8 \end{pmatrix} = \begin{pmatrix} 7 + 2 \cdot 8 \\
3 \cdot 7 + 4 \cdot 8 \\
5 \cdot 7 + 6 \cdot 8 \end{pmatrix} = \begin{pmatrix} 23 \\
53 \\
83 \end{pmatrix}
\]

matrix $\times$ column vector: \[
\begin{pmatrix} r_1 \\
\vdots \\
r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\
\vdots \\
r_m b \end{pmatrix}
\]
A matrix equation is an equation $Ax = b$ where $A$ is a matrix and $b$ is a vector. So $x$ is a vector of variables.

$A$ is an $m \times n$ matrix if it has $m$ rows and $n$ columns. What sizes must $x$ and $b$ be?

Example:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$

Rewrite this equation as a vector equation, a system of linear equations, and an augmented matrix.

We will go back and forth between these four points of view over and over again. You need to get comfortable with this.
Solutions to Linear Systems vs Spans

Say that

\[
A = \begin{pmatrix}
| & | & \\
v_1 & v_2 & \cdots & v_n \\
| & | & \\
\end{pmatrix}.
\]

Fact. \(Ax = b\) has a solution \(\iff\) \(b\) is in the span of columns of \(A\)

Why?

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
\end{pmatrix} =
\begin{pmatrix}
5 \\
6 \\
\end{pmatrix}
\]

A solution means \(x(1) + y(2) = (5, 6)\) has a soln.

Again this is a basic fact we will use over and over and over and over.
Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution $\iff b$ is in the span of columns of $A$

Examples:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
3 \\
5 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
3 \\
0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 3 \\
0 & 0 & 5 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix}
\]

- \text{Not Span of Cols of } A
- \text{Inconsistent of } A
- \text{b not in xy-plane}
- \text{Consistent}
- \text{(2 3 0) is in xy-plane}
- \text{Solution: (2 3)}

Compare with color mixing problem.
Is a given vector in the span?

Fact. $Ax = b$ has a solution $\iff b$ is in the span of columns of $A$

algebra $\iff$ geometry

Is $(9,10,11)$ in the span of $(1,3,5)$ and $(2,4,6)$?

yes. Find $x$ & $y$!

\[ \begin{align*}
\begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \\ 4 \\ 5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 4 \\ 6 \\ 5 \\ 4 \\ 6 \end{bmatrix} y &= \begin{bmatrix} 9 \\ 10 \\ 11 \\ 10 \\ 10 \\ 11 \end{bmatrix} \\
\begin{bmatrix} 1 & 2 & 9 \\ 0 & -2 & -17 \\ 0 & -4 & -34 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 9 \\ 0 & -2 & -17 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 9 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*} \]

row reduce. pivot in last col. $\implies$ no
Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. (0, 1, 2) is in the span of (3, 3, 4), (0, 10, 20), (0, −1, −2)
2. (0, 1, 2) is in the span of (3, 3, 4), (0, 1, 0), (0, 0, \( \sqrt{2} \))
3. (0, 1, 2) is in the span of (3, 3, 4), (0, 5, 7), (0, 6, 8)
4. (0, 1, 2) is in the span of (5, 7, 0), (6, 8, 0), (3, 3, 4)
Pivots vs Solutions

Theorem. Let $A$ be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all $b$
2. The span of the columns of $A$ is $\mathbb{R}^m$
3. $A$ has a pivot in each row

Why?

1 is same as 2: 1 means all $b$ lie in span of cols $\sim$ span of $A$

1 is same as 3:

If $A$ didn't have pivot each row

More generally, if you have some vectors and you want to know the dimension of the span, you should row reduce and count the number of pivots.
Properties of the Matrix Product $Ax$

c = real number, $u, v = $ vectors,

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

**Application.** If $u$ and $v$ are solutions to $Ax = 0$ then so is every element of $\text{Span}\{u, v\}$. 
Guiding questions

Here are the guiding questions for the rest of the chapter:

1. What are the solutions to $Ax = \emptyset$?

2. For which $b$ is $Ax = b$ consistent?

These are two separate questions!
Summary of Section 2.3

- Two ways to multiply a matrix times a column vector:

\[
\begin{pmatrix}
  r_1 \\
  \vdots \\
  r_m
\end{pmatrix}
\begin{pmatrix} b \\
  \vdots \\
  r_mb
\end{pmatrix}
\]

OR

\[
\begin{pmatrix}
  x_1 & x_2 & \cdots & x_n
\end{pmatrix}
\begin{pmatrix} b_1 \\
  \vdots \\
  b_n
\end{pmatrix}
= \begin{pmatrix} b_1 x_1 & \cdots & b_n x_n \end{pmatrix}
\]

- Linear systems, augmented matrices, vector equations, and matrix equations are all equivalent.

- Fact. \( Ax = b \) has a solution \( \iff b \) is in the span of columns of \( A \)

- Theorem. Let \( A \) be an \( m \times n \) matrix. The following are equivalent.
  1. \( Ax = b \) has a solution for all \( b \)
  2. The span of the columns of \( A \) is \( \mathbb{R}^m \)
  3. \( A \) has a pivot in each row
Typical exam questions

• If \( A \) is a \( 3 \times 5 \) matrix, and the product \( Ax \) makes sense, then which \( \mathbb{R}^n \) does \( x \) lie in?

• Rewrite the following linear system as a matrix equation and a vector equation:

\[
x + y + z = 1
\]

• Multiply:

\[
\begin{pmatrix}
0 & 2 \\
0 & 4 \\
5 & 0
\end{pmatrix}
\begin{pmatrix}
3 \\
2
\end{pmatrix}
\]

• Which of the following matrix equations are consistent?

\[
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
2 \\
3 \\
2
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
2 \\
3 \\
3
\end{pmatrix}
\]

(And can you do it without row reducing?)