Announcements Sep 23

- WeBWorK on Section 2.6 due Thursday night
- Quiz on Section 2.6 Friday 8 am - 8 pm EDT
- My Office Hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Regular Studio on Friday
- Second Midterm Friday Oct 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- Tutoring: http://tutoring.gatech.edu/tutoring
- PLUS sessions: http://tutoring.gatech.edu/plus-sessions
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: https://counseling.gatech.edu
Section 2.7

Bases
Bases

$V = \text{subspace of } \mathbb{R}^n$

A basis for $V$ is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

1. $V = \text{Span}\{v_1, \ldots, v_k\}$
2. $v_1, \ldots, v_k$ are linearly independent

Equivalently, a basis is a *minimal spanning set*, that is, a spanning set where if you remove any one of the vectors you no longer have a spanning set.

Q. What is one basis for $\mathbb{R}^2$? $\mathbb{R}^n$? How many bases are there?
Dimension

$V = \text{subspace of } \mathbb{R}^n$

$\dim(V) = \text{dimension of } V = k = \text{the number of vectors in the basis}$

(What is the problem with this definition of dimension?)
Basis theorem

Basis Theorem
If $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$, then

- any $k$ linearly independent vectors of $V$ form a basis for $V$
- any $k$ vectors that span $V$ form a basis for $V$

In other words if a set has two of these three properties, it is a basis:

- spans $V$, linearly independent, $k$ vectors

We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.
Section 2.9
The rank theorem
Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\text{Col}(A)$:
Rank Theorem

\[ \text{rank}(A) = \dim \text{Col}(A) = \# \text{ pivot columns} \]
\[ \text{nullity}(A) = \dim \text{Nul}(A) = \# \text{ nonpivot columns} \]

Rank Theorem. \( \text{rank}(A) + \text{nullity}(A) = \# \text{cols}(A) \)

This ties together everything in the whole chapter: rank \( A \) describes the \( b \)'s so that \( Ax = b \) is consistent and the nullity describes the solutions to \( Ax = 0 \). So more flexibility with \( b \) means less flexibility with \( x \), and vice versa.

Example. \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \)
About names

Again, why did we need all these vocabulary words? One answer is that the rank theorem would be harder to understand if it was:

The size of a minimal spanning set for the set of solutions to $Ax = 0$ plus the size of a minimal spanning set for the set of $b$ so that $Ax = b$ has a solution is equal to the number of columns of $A$.

Compare to: $\text{rank}(A) + \text{nullity}(A) = n$

“A common concept in history is that knowing the name of something or someone gives one power over that thing or person.” –Loren Graham
http://philoctetes.org/news/the_power_of_names_religion_mathematics
Section 2.9 Summary

- **Rank Theorem.** \( \text{rank}(A) + \dim \text{Nul}(A) = \#\text{cols}(A) \)
Typical exam questions

- Suppose that $A$ is a $5 \times 7$ matrix, and that the column space of $A$ is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that $A$ is a $5 \times 7$ matrix, and that the column space of $A$ is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that $A$ is a $5 \times 7$ matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a $3 \times 2$ matrix so that the column space and the null space are both lines.
- True/false. There is a $2 \times 3$ matrix so that the column space and the null space are both lines.
- True/false. Suppose that $A$ is a $6 \times 2$ matrix and that the column space of $A$ is 5 dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some $b$ in $\mathbb{R}^6$?
Sections 3.1
Matrix Transformations
Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$
From matrices to functions

Let $A$ be an $m \times n$ matrix.

We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(v) = Av$$

This is called a matrix transformation.

The domain of $T$ is $\mathbb{R}^n$.

The co-domain of $T$ is $\mathbb{R}^m$.

The range of $T$ is the set of outputs: $\text{Col}(A)$

This gives us another point of view of $Ax = b$.

Example: range of $f(x) = x^2$ in Calc 1 is $[0, \infty)$. Co-domain is $\mathbb{R}$. 
Example

Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \), \( u = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \), \( b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \).

What is \( T(u) \)?

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}
\]

Find \( v \) in \( \mathbb{R}^2 \) so that \( T(v) = b \)

\[
v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}
\]

Find a vector in \( \mathbb{R}^3 \) that is not in the range of \( T \).

any vector with different \( 1^{st} \) and \( 3^{rd} \) entries.

since \[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} x \\ 4 \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x+y \end{pmatrix}
\]
Square matrices

For a square matrix we can think of the associated matrix transformation

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

as doing something to \( \mathbb{R}^n \).

Example. The matrix transformation \( T \) for

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

What does \( T \) do to \( \mathbb{R}^2 \)?

Reflection about \( y \)-axis

Input: \( (x, y) \)

Output: \( (-x, y) \)
Square matrices

What does each matrix do to $\mathbb{R}^2$?

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

Input: \((x, y)\)

Output: \((y, x)\)

(Reflection about \(y=x\).

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix}
\]

Input: \((x, y)\)

Output: \((x, 0)\)

(Orthogonal projection to \(x\)-axis)

\[
\begin{pmatrix}
3 & 0 \\
0 & 3 \\
\end{pmatrix}
\]

Input: \((x, y)\)

Output: \((3x, 3y)\)

(Dilation by 3)

What is the range in each case?

Range: $\mathbb{R}^2$
Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?
Square matrices

What does each matrix do to $\mathbb{R}^2$?

*Hint: if you can’t see it all at once, see what happens to the $x$- and $y$-axes.*

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

Input: \( (x, y) \)
Output: \( (x+y, y) \)

\[
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\]

\( \theta = \text{some fixed number} \)

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

Input: \( (1, 0) \)
Output: \( (\cos \theta, \sin \theta) \)

Input: \( (0, 1) \)
Output: \( (-\sin \theta, \cos \theta) \)
Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

1. Matrix:
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
Input: $(x, y, z)$
Output: $(x, y, 0)$

2. Matrix:
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Input: $(x, y, z)$
Output: $(x, -y, z)$

3. Matrix:
\[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Input: $(x, y, z)$
Output: $(0, y, z)$

Orthogonal Projection to $xy$ plane

Range: $xy$-plane

Reflection about $xz$-plane

Range: $\mathbb{R}^3$

Rotation by $90^\circ$ about $z$-axis

Range: $\mathbb{R}^3$
Section 3.1 Summary

- If $A$ is an $m \times n$ matrix, then the associated matrix transformation $T$ is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\text{Col}(A)$.

- If $A$ is $n \times n$ then $T$ does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation.

Key example: Rabbits

- $\frac{1}{2}$ first years survive
- $\frac{1}{2}$ second yrs survive
- 2nd have 6 babies
- 3rd yrs have 8 babies

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

First day slides

First day slides

Input: Population in a given year (#1st years, #2nd, #3rd)

Output: Population for following year

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} F \\ S \\ F \end{pmatrix} = \begin{pmatrix} 6S + 8T \\ \frac{1}{2}F \\ \frac{1}{2}S \end{pmatrix}$$

No!
Typical exam questions

- What does the matrix \(
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\) do to \(\mathbb{R}^2\)?

- What does the matrix \(
\begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
-1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}
\) do to \(\mathbb{R}^2\)?

- What does the matrix \(
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\) do to \(\mathbb{R}^3\)?

- What does the matrix \(
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\) do to \(\mathbb{R}^2\)?

- True/false. If \(A\) is a matrix and \(T\) is the associated matrix transformation, then the statement \(Ax = b\) is consistent is equivalent to the statement that \(b\) is in the range of \(T\).

- True/false. There is a matrix \(A\) so that the domain of the associated matrix transformation is a line in \(\mathbb{R}^3\).
Sections 3.2

One-to-one and onto transformations
Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto
One-to-one

\( T : \mathbb{R}^n \to \mathbb{R}^m \) is one-to-one if each \( b \) in \( \mathbb{R}^m \) is the output for at most one \( v \) in \( \mathbb{R}^n \).

In other words: different inputs have different outputs.

**Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:

- \( T \) is one-to-one
- the columns of \( A \) are linearly independent
- \( Ax = 0 \) has only the trivial solution
- \( A \) has a pivot in each column
- the range of \( T \) has dimension \( n \)

What can we say about the relative sizes of \( m \) and \( n \) if \( T \) is one-to-one?

Draw a picture of the range of a one-to-one matrix transformation \( \mathbb{R} \to \mathbb{R}^3 \).
Onto

\( T : \mathbb{R}^n \to \mathbb{R}^m \) is onto if the range of \( T \) equals the codomain \( \mathbb{R}^m \), that is, each \( b \) in \( \mathbb{R}^m \) is the output for at least one input \( v \) in \( \mathbb{R}^m \).

**Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:

- \( T \) is onto
- the columns of \( A \) span \( \mathbb{R}^m \)
- \( A \) has a pivot in each row
- \( Ax = b \) is consistent for all \( b \) in \( \mathbb{R}^m \)
- the range of \( T \) has dimension \( m \)

What can we say about the relative sizes of \( m \) and \( n \) if \( T \) is onto?

Give an example of an onto matrix transformation \( \mathbb{R}^3 \to \mathbb{R} \).
One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

\[
\begin{pmatrix}
1 & 0 & 7 \\
0 & 1 & 2 \\
0 & 0 & 9
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 \\
1 & 1 \\
2 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 1 & 1
\end{pmatrix} \quad \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\]
One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \text{ reflection}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \text{ projection}
\]

\[
\begin{pmatrix}
3 & 0 \\
0 & 3
\end{pmatrix} \text{ scaling}
\]

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \text{ shear}
\]

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \text{ rotation}
\]