What does \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) do to this letter F?

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}
\]

"shear"
Announcements Sep 28

- WeBWorK on Sections 2.7, 2.9, 3.1 due Thursday night
- Quiz on Section 2.7, 2.9, 3.1 Friday 8 am - 8 pm EDT
- My Office Hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Second Midterm Friday Oct 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- Tutoring: http://tutoring.gatech.edu/tutoring
- PLUS sessions: http://tutoring.gatech.edu/plus-sessions
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: https://counseling.gatech.edu
Sections 3.1
Matrix Transformations

Chapter 3: reframing Chaps 1 & 2 in terms of algebra
From matrices to functions

Let $A$ be an $m \times n$ matrix.

We define a function $T : \mathbb{R}^n \to \mathbb{R}^m$

$$T(v) = Av$$

This is called a matrix transformation.

The domain of $T$ is $\mathbb{R}^n$.

The co-domain of $T$ is $\mathbb{R}^m$.

The range of $T$ is the set of outputs: $\text{Col}(A)$

This gives us another point of view of $Ax = b$
Why are we learning about matrix transformations?

Sample applications:

- Cryptography (Hill cypher)
- Computer graphics (Perspective projection is a linear map!)
- Aerospace (Control systems - input/output)
- Biology
- Many more!
Applications of Linear Algebra

**Biology:** In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

These relations can be represented using a matrix.

\[
\begin{pmatrix}
0 & 6 & 8 \\
\frac{1}{2} & 0 & 10 \\
0 & \frac{1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
4 \\
6 \\
7
\end{pmatrix}
= 
\begin{pmatrix}
92 \\
2 \\
3
\end{pmatrix}
\]

How does this relate to matrix transformations?
Section 3.2

One-to-one and onto transformations
Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto

In Calculus:

One-to-one

- Horizontal line test: each horizontal line crosses graph at most 1 pt.
- Each input has one output

Onto

- Each horizontal line crosses graph at least 1 pt.
- All possible outputs are actual outputs
- Codomain = range

Example: \( f(x) = x^2 \) not onto.
- \(-1\) is not in the range
One-to-one

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each $b$ in $\mathbb{R}^m$ is the output for at most one $v$ in $\mathbb{R}^n$.

In other words: different inputs have different outputs.

Do not confuse this with the definition of a function, which says that for each input $x$ in $\mathbb{R}^n$ there is at most one output $b$ in $\mathbb{R}^m$. 
One-to-one

\( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one-to-one if each \( b \) in \( \mathbb{R}^m \) is the output for at most one \( v \) in \( \mathbb{R}^n \).

**Theorem.** Suppose \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:

- \( T \) is one-to-one
- the columns of \( A \) are linearly independent
- \( Ax = 0 \) has only the trivial solution
- \( A \) has a pivot in each column
- the range of \( T \) has dimension \( n \)

\[ \text{Col}(A) \]

What can we say about the relative sizes of \( m \) and \( n \) if \( T \) is one-to-one?

\[ m \geq n \text{ tall or square, not wide} \]

Draw a picture of the range of a one-to-one matrix transformation \( \mathbb{R} \rightarrow \mathbb{R}^3 \).
Onto

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of $T$ equals the codomain $\mathbb{R}^m$, that is, each $b$ in $\mathbb{R}^m$ is the output for at least one input $v$ in $\mathbb{R}^m$. 
Onto

\[ T : \mathbb{R}^n \to \mathbb{R}^m \text{ is onto} \] if the range of \( T \) equals the codomain \( \mathbb{R}^m \), that is, each \( b \) in \( \mathbb{R}^m \) is the output for at least one input \( v \) in \( \mathbb{R}^m \).

**Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:

- \( T \) is onto
- the columns of \( A \) span \( \mathbb{R}^m \)
- \( A \) has a pivot in each row
- \( Ax = b \) is consistent for all \( b \) in \( \mathbb{R}^m \)
- the range of \( T \) has dimension \( m \)

\[ \text{Col}(A) = \mathbb{R}^m \]

What can we say about the relative sizes of \( m \) and \( n \) if \( T \) is onto?

\[ m \leq n \quad \text{wide or square, not tall} \]

Give an example of an onto matrix transformation \( \mathbb{R}^3 \to \mathbb{R}^2 \).

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

projection
One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

\[
\begin{pmatrix}
1 & 0 & 7 \\
0 & 1 & 2 \\
0 & 0 & 9
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
2 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}
\]

one-to-one ✓ ✓ ✓ ✓
onto ✓ ✓ ✓ ✓

By the way: onto \iff \text{rows lin ind.}
One-to-one and Onto

Which of the previously-studied matrix transformations of \( \mathbb{R}^2 \) are one-to-one? Onto?

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
reflection about \( y = x \)

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]
projection onto \( x\)-axis

\[
\begin{pmatrix}
3 & 0 \\
0 & 3
\end{pmatrix}
\]
scaling

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]
shear

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
rotation
Which are one to one / onto?

Poll

Which give one to one-to-one / onto matrix transformations?

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
1 & -1 & 2 \\
-2 & 2 & -4
\end{pmatrix}
\]

\(f(x) = x^2\) is one-to-one

Like \(f(x) = x^2\)

Inputs: 2, -2

Same output: 4

Not one-to-one

Find 2 inputs with same output for

Inputs: \((0, 0), (-2, 0), (-1, 0)\)

Same Output \(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\)

Not one-to-one
Consider the robot arm example from the book.

There is a natural function $f$ here (not a matrix transformation). The input is a set of three angles and the co-domain is $\mathbb{R}^2$. Is this function one-to-one? Onto?
Summary of Section 3.2

- \( T : \mathbb{R}^n \to \mathbb{R}^m \) is **one-to-one** if each \( b \) in \( \mathbb{R}^m \) is the output for at most one \( v \) in \( \mathbb{R}^n \).

- **Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:
  - \( T \) is one-to-one
  - the columns of \( A \) are linearly independent
  - \( Ax = 0 \) has only the trivial solution
  - \( A \) has a pivot in each column
  - the range has dimension \( n \)

- \( T : \mathbb{R}^n \to \mathbb{R}^m \) is **onto** if the range of \( T \) equals the codomain \( \mathbb{R}^m \), that is, each \( b \) in \( \mathbb{R}^m \) is the output for at least one input \( v \) in \( \mathbb{R}^n \).

- **Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:
  - \( T \) is onto
  - the columns of \( A \) span \( \mathbb{R}^m \)
  - \( A \) has a pivot in each row
  - \( Ax = b \) is consistent for all \( b \) in \( \mathbb{R}^m \).
  - the range of \( T \) has dimension \( m \).
Typical exam questions

- True/False. It is possible for the matrix transformation for a $5 \times 6$ matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by projection to the $yz$-plane is onto.
- True/False. The matrix transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by rotation by $\pi$ is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^3$? If so, write one down, if not explain why not.
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^3$? If so, write one down, if not explain why not.
Section 3.3

Linear Transformations
Section 3.3 Outline

- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Find the matrix for a linear transformation
Linear transformations

A function \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation if

- \( T(u + v) = T(u) + T(v) \) for all \( u, v \) in \( \mathbb{R}^n \).
- \( T(cv) = cT(v) \) for all \( v \) in \( \mathbb{R}^n \) and \( c \) in \( \mathbb{R} \).

First examples: matrix transformations.

\[
A(u + v) = Au + Av
\]
\[
A(cu) = cAu
\]
Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if

- $T(u + v) = T(u) + T(v)$ for all $u, v$ in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all $v$ in $\mathbb{R}^n$ and $c$ in $\mathbb{R}$.

Notice that $T(0) = 0$. *Why?*

We have the standard basis vectors for $\mathbb{R}^n$:

- $e_1 = (1, 0, 0, \ldots, 0)$
- $e_2 = (0, 1, 0, \ldots, 0)$
- $\vdots$

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. *Why?*

In engineering, this is called the principle of superposition.
Linear transformations are matrix transformations

**Theorem.** Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix $A$ so that

$$T(v) = Av$$

for all $v$ in $\mathbb{R}^n$.

The matrix for a linear transformation is called the **standard matrix**.
Linear transformations are matrix transformations

**Theorem.** Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix}$$

Why? Notice that $Ae_i = T(e_i)$ for all $i$. Then it follows from linearity that $T(v) = Av$ for all $v$. 
The identity

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$T(v) = v$$

What is the standard matrix?

This standard matrix is called $I_n$ or $I$. 
Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the function given by:

$$T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + y \\ y \\ x - y \end{array} \right)$$

What is the standard matrix for $T$?

In fact, a function $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables, no constant terms).
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the $x$-direction and 3 in the $y$-direction, and then reflects over the line $y = x$. 
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the $y$-axis and then rotates counterclockwise by $\pi/2$. 
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane.
Discussion

Find a matrix that does this.
Summary of 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u + v) = T(u) + T(v)$ for all $u, v$ in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and $c$ in $\mathbb{R}$.

- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).

- The standard matrix for a linear transformation has its $i$th column equal to $T(e_i)$.