Math 1553 Worksheet §§3.4, 3.5, 3.6

1. If $A$ is a $3 \times 5$ matrix and $B$ is a $3 \times 2$ matrix, which of the following are defined?
   a) $A - B$
   b) $AB$
   c) $A^T B$
   d) $B^T A$
   e) $A^2$

2. Consider the following linear transformations:
   $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ $T$ projects onto the $xy$-plane, forgetting the $z$-coordinate
   $U: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $U$ rotates clockwise by 90°
   $V: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $V$ scales the $x$-direction by a factor of 2.

Let $A, B, C$ be the matrices for $T, U, V$, respectively.
   a) Compute $A, B, C$.
   b) Compute the matrix for $V \circ U \circ T$.
   c) Compute the matrix for $U \circ V \circ T$.
   d) Describe $U^{-1}$ and $V^{-1}$, and compute their matrices.
3. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.
   
a) If \( A \) is an \( m \times n \) matrix and \( B \) is an \( n \times p \) matrix, then each column of \( AB \) is a linear combination of the columns of \( A \).

b) If \( A \) and \( B \) are \( n \times n \) and both are invertible, then the inverse of \( AB \) is \( A^{-1}B^{-1} \).

c) If \( A^T \) is not invertible, then \( A \) is not invertible.

d) If \( A \) is an \( n \times n \) matrix and the equation \( Ax = b \) has at least one solution for each \( b \) in \( \mathbb{R}^n \), then the solution is unique for each \( b \) in \( \mathbb{R}^n \).

e) If \( A \) and \( B \) are invertible \( n \times n \) matrices, then \( A+B \) is invertible and \( (A+B)^{-1} = A^{-1} + B^{-1} \).

f) If \( A \) and \( B \) are \( n \times n \) matrices and \( ABx = 0 \) has a unique solution, then \( Ax = 0 \) has a unique solution.

4. Suppose \( A \) is an invertible \( 3 \times 3 \) matrix with the following equations hold. Find \( A \).

\[
A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]