Math 8803:
Characteristic Classes
of Vector Bundles
and Surface Bundles

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Theory of Characteristic classes:

**Bundles over **$B$** → $H^*(B)$**

so as to distinguish bundles, e.g.

○ vs ○

This course: Vector bundles, surface bundles.

**VECTOR BUNDLES**

$E$ $p^*(B) =$ fiber $\leftarrow$ Struct. of vector space $V$.

$\begin{array}{c}
B = \text{base} \\
p : B \text{ covered by } U \ \text{s.t.} \\
B \times V \rightarrow p^{-1}(U) \\
\end{array}$

Important because; smooth manifolds have tangent bundles, submanifolds have normal bundles.

e.g. can distinguish two smooth structures on a manifold if we can distinguish their tangent bundles using characteristic classes.

Thm (Milnor) $\exists$ exotic 7-spheres.
**Characteristic Classes**

A char. class for vect. bundles is a function
\[ \chi: \{ V\text{-bundles over } B \} \rightarrow H^k(B; G) \]
for fixed \( V, k, G \) (\( B \) allowed to vary!)
that is natural:
\[ \chi(f^*(E)) = f^* \chi(E) \]

**Euler Class**

Take \( V = \mathbb{R}^n, k=n, G=\mathbb{Z}, \) restrict to oriented bundles.

\[ \Rightarrow \text{ Euler class } e. \]

\[ B = M, E = TM \Rightarrow e(TM) \in H^n(M; \mathbb{Z}) \cong \mathbb{Z} \]
\[ \chi(M). \]

Euler char is a char. class. It has many interpretations, e.g:

1. **Combinatorial**: \( \chi(M) = \sum (-1)^i (\# \text{ } i\text{-cells}) \)
2. **Geometric**: \( \chi(M) = \frac{1}{\omega_{n-1}} \int_M \chi(x) \text{ dvol}_M \)
3. **Homological**: \( \chi(M) = \sum (-1)^i \text{ rank } H_i(M; \mathbb{Z}) \)
4. **Cohomological**: \( \chi(M) = \text{ self-intersection of } M \text{ in } TM. \)

(4) implies \( \chi(M) \) is obstruction to nonvanishing vector field (recall Thurston's proof).
Grassmann Manifolds

Euler class is so beautiful, we want to find all other char. classes.

\( G_n = \text{space of } n\text{-planes in } \mathbb{R}^\infty. \)
\( E_n = \text{canonical bundle over } G_n: \)
\((n\text{-plane in } \mathbb{R}^\infty, \text{ vector in that plane}) \subseteq G_n \times \mathbb{R}^\infty.\)

We will show:
\[
\{ \text{\( \mathbb{R}^n \)-bundles over } B \}/\text{isomorp.} \iff \{ \text{maps } B \rightarrow G_n \}/\text{homotopy},
\]
\[ f^*(E_n) \iff f \]

This gives:
\[
\{ \text{char. classes for } \mathbb{R}^n\text{-bundles} \}/\text{G-coeff} \iff H^*(G_n; G).
\]

Goal: compute the latter.

If we care about:
- complex bundles \( \rightarrow G_n(G) \)
- oriented real bundles \( \rightarrow G_n^\mathbb{R} \)
**Stiefel-Whitney Classes**

We will show: $H^k(Gn; \mathbb{Z}_2) \cong \mathbb{Z}_2 [W_1, \ldots, W_n]$  
$W_i$ called $i^{th}$ SW class.

$W_i$ is very concrete $\in H^i(B; \mathbb{Z}_2) \cong \text{Hom}(H_i(B; \mathbb{Z}_2); \mathbb{Z})$  
$H$ records whether the bundle is orientable over $\text{an element of } H_i$.

$W_i$ = obstruction to finding $n-k+1$ indep. sections  
over the $i$-skeleton of $B$.

**Thm (Thom).** Two manifolds are cobordant iff their SW numbers of their tangent bundles are equal.

**Other Characteristic Classes**

<table>
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**Surface Bundles**

\[ S_g = \cdots \]

\[ S_g \text{-bundle} \quad E \quad \quad \quad \quad \quad \quad \quad p \downarrow \]

\[ B \quad \quad \quad p^{-1}(U) = U \times S_g \]

Important class of manifolds (also, they are the next-simplest bundles).

Characteristic class

\[ \chi : \left\{ \text{oriented } S_g \text{-bundles} \right\}_{\text{over } B} / \text{isom.} \rightarrow H^*(B; G) \]

naturality \[ \chi(f^*(E)) = f^*(\chi(E)) \]

Classifying space

\[ \left\{ \text{oriented } S_g \text{-bundles} \right\}_{\text{over } B} / \text{isom.} \leftrightarrow \left\{ \text{maps } B \rightarrow \text{BHomeo}^+(S_g) \right\} / \text{hom.} \]

\[ \text{BHomeo}^+(S_g) = \text{space of } S_g \text{-submanifolds of } \mathbb{R}^\infty \]

\[ = K(\text{MCG}(S_g), 1) \]

So: char. classes for orient. \( S_g \)-bundles \( \leftrightarrow H^*(\text{MCG}(S_g); G) \).

We do not have a full list, but

\[ e_i \in H^{2i}(\text{MCG}(S_g); \mathbb{Z}) \quad \text{Morita-Mumford-Miller classes} \]

\[ \text{generate } H^*(\text{MCG}(S_g)) \text{ stably} \quad \text{(Madson-Weiss)}. \]
Morita’s Theorem

\[ \pi : \text{Diff}^+(S^g) \rightarrow \text{MCG}(S^g) \text{ has no section } g \gg 0. \]

Proof: \( e_3 \neq 0 \), \( \pi^*(e_3) = 0 \).

Odd MMM classes are geometric.

\[ e_1 \in H^2(B; \mathbb{Z}) \quad \text{WLOG: } B = \text{surface}, \]
\[ \Rightarrow E = 4\text{-manifold } M \]

Hirzebruch: \( E_1(M) = \tau(M) \) signature.

But \( \tau \) (hence \( e_1 \)) ignores bundle structure,
even though \( e_1 \) defined via bundle structure.
Say \( e_1 \) is geometric.

Thm (Church–Farb–Thibault) \( e_{2i+1} \) is geometric.

E.g. \( J \) \( S^1 \)-bundle over \( S^4 \approx S^4 \) bundle over \( S^2 \).

Pf that \( e_1 \) is geometric: \( e_1(E) = p_1(M) \leftarrow \text{1st Pontryagin class} \]
\[ = \tau(M) \text{ (Hirzebruch).} \]
Vector Bundles

Fix a vector space V

\[ V \rightarrow E \]
\[ p \uparrow \]
\[ B \]

1. Fibers \( p^{-1}(b) \) have structure of \( V \).
2. \( B \) covered by \( U \) s.t. \( E \)
   \[ p^{-1}(U) \rightarrow U \times V \] homeo resp.
   
   local
   trivialization

Examples

1. Trivial bundle \( E = B \times V \).
2. Moebius bundle over \( S^1 \).
3. Tangent bundle to a smooth manifold \( M \)

\[ TM = \{(x,v) : v \in T_xM\} \]
\[ p(x,v) = x \]

v.s. structure:
\[ k_1(x,v_1) + k_2(x,v_2) = (x, k_1 v_1 + k_2 v_2) \]

By defn, \( M \) locally diff to \( U \subseteq \mathbb{R}^n \) open.
So suffices to show \( TU \) locally trivial. easy

4. Normal bundle to \( M \hookrightarrow N \)

Locally: \( \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k} \) (Tubular nbhd thm).
5. **Canonical bundle over $\mathbb{RP}^n$**

$\mathbb{RP}^n = \text{space of lines in } \mathbb{R}^{n+1} \cong S^n/\text{antipode}$

Canonical line bundle: $\{(l, v) : v \in l\}$

Local trivialization near $l$: orthog. proj. to $l$ in $\mathbb{R}^{n+1}$

\[ (l', v) \mapsto (l', \text{proj}_l(v)) \in U \times l. \]

Allow $n = \infty$.

6. **Orthogonal complement to 5**

$E^\perp = \{(l, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \perp l\}$

Again, orthog proj gives local trivialization.

\[ \text{Q. } E^\perp \cong \mathbb{TP}^n? \]

7. **Grassmann manifold**

$G_n = \text{space of } n\text{-planes in } \mathbb{R}^\infty \text{ thru } 0$.

$E_n = \{(P, v) \in G_n \times \mathbb{R}^\infty : v \in P\}$

& $E_n^\perp = \{(P, v) \in G_n \times \mathbb{R}^\infty : v \perp P\}$

8. **Vertical bundle of surface bundle**

\[ \text{Char. classes for surface bundles defined in terms of char. classes for these vector bundles.} \]
ISOMORPHISM

\[ p_1 : E_1 \to B \] is isomorphic to \[ p_2 : E_2 \to B \]
if \exists \text{ homeo } h : E_1 \to E_2 \text{ s.t. } h|_{p_1^{-1}(b)} \text{ is a v.s. } \cong \text{ to } p_2^{-1}(b).

\[
\text{N.B. } \infty \neq \text{triv. Möb}
\]

& bundles over different spaces can't be isomorphic(!)

EXAMPLES

1. \[ \text{NS}^n \cong S^n \times \mathbb{R} \]
   via \( (x, tx) \mapsto (x, t) \)

2. \[ TS' \cong S' \times \mathbb{R} \]
   via \( (z, izt) \mapsto (z, t) \)

We say \( S' \) is parallelizable.

Q. Which manifolds are parallelizable? \( S^2 \)?
   (All 3-manifolds!)

3. Can cn. line bundle over \( TP' \cong \text{Möbius bundle over } TP' \)
   after traveling around base, fibers get flipped:

Q. Is \( TTP^n \cong E^4 \)?
A section of \( p: E \to B \) is \( s: B \to E \) s.t. \( p \circ s = \text{id} \).

e.g. 0-section

Some bundles have non-vanishing sections, some do not.

For example: A section of \( TM \) is a vector field on \( M \).

We showed nonvan vect field \( \Rightarrow \chi(M) = 0 \).

So \( \chi(M) \neq 0 \Rightarrow TM \) has no nonvan. sec.

\( \chi(S^n) = 2 \ n \) even.

Can show \( S^n \) has nonvan. vect field \( n \) odd.

FACT: An \( n \)-dim bundle is trivial \( \iff \) it has \( n \) sections \( s \) that are lin. ind. over each point of \( B \).

\( \Rightarrow \) obvious

\( \Leftarrow \) there is a contin. map

\[ B \times \mathbb{R}^n \to E \]

\[ (b, t_1, \ldots, t_n) \mapsto \sum t_i S_i(b) \]

Clearly isom. on fibers

need to show inverse is continuous

follows from: inversion of matrices is continuous.

Spheres: \( TS^1 \) trivial by \( S_1(z) = iz \)

\( TS^3 \) trivial by \( S_1(z) = iz, \ S_2(z) = jz, \ S_3(z) = kz \)

\( TS^7 \) trivial by similar construction w/ octonians.

(all other \( TS^n \) nontrivial!!)
**Direct Sum**

\[ p_1 : E_1 \to B, \quad p_2 : E_2 \to B \quad \to \]

\[ E_1 \oplus E_2 = \left\{ (v_1, v_2) \in E_1 \times E_2 : \quad p_1(v_1) = p_2(v_2) \right\} \]

\[ p : E_1 \oplus E_2 \to B \]

\[ (v_1, v_2) \to p(v_1) \]

\( E_1 \oplus E_2 \) a vector bundle because

1. products of vb's are vb's
2. restrictions of vb's are vb's.

\( E_1 \oplus E_2 \) is restriction of \( E_1 \times E_2 \) to diagonal \( B \subset B \times B \).

Trivial \( \oplus \) trivial = trivial but

Nontrivial \( \oplus \) trivial can be trivial!

e.g. \( TS^n \oplus NS^n \) trivial. Say \( TS^n \) stably trivial.

\( \sim \) trivial

Also: \( E \oplus E^\perp \to TP^n \) trivial via \( \ell(v, w) \to (\ell, v+w) \)

n=1 case: \( \text{Möbius} \oplus \text{Möbius} = \text{trivial} \)

A useful exercise, related to last example: Show there are exactly two \( \mathbb{R}^n \) bundles over \( S^1 \). Similarly, exactly two \( S^1 \)-bundles over \( S^1 \).
EXAMPLE. $T\mathbb{RP}^n$ stably isom. to $\bigoplus_i E$

Start with $T\mathbb{S}^n \oplus NS^n \cong S^n \times \mathbb{R}^{n+1}$

Quotient by $(x,v) \sim (-x,-v)$ on both sides.

$T\mathbb{S}^n/\sim \cong T\mathbb{RP}^n$ since $(x,v) \mapsto (-x,-v)$ is map on $T\mathbb{S}^n$ induce by $x \mapsto -x$.

$NS^n/\sim \cong \mathbb{RP}^n \times \mathbb{R}$ via the section $x \mapsto (x,x)$

Claim: $(S^n \times \mathbb{R}^{n+1})/\sim \cong \bigoplus_{i=1}^{n+1} E$

First, $\sim$ preserves factors, so

$$(S^n \times \mathbb{R}^{n+1})/\sim \cong \bigoplus_{i=1}^{n+1} (S^n \times \mathbb{R})/\sim$$

But $(S^n \times \mathbb{R})/\sim \cong E$, as

Using quaternions, $T\mathbb{RP}^3 \cong T\mathbb{RP}^3 \times \mathbb{R}^3$

As above, $T\mathbb{RP}^3 \oplus$ trivial line bundle $\cong \mathbb{RP}^3 \times \mathbb{R}^4$

As above, $T\mathbb{RP}^3 \oplus$ trivial line bundle $\cong \bigoplus_i E$

$\Rightarrow \bigoplus_{i=1}^4 E \cong \mathbb{RP}^3 \times \mathbb{R}^4$. 
Next Goal

Prop. \( B = \text{compact Hausdorff} \)

\[ \forall E \rightarrow B \exists E' \rightarrow B \text{ s.t. } E \oplus E' \text{ trivial.} \]

Step 1. Inner Products

Inner product on \( V \): pos. def. symm. bilinear form.

Inner product on \( E \): map \( E \oplus E \rightarrow \mathbb{R} \) restricting to inner prod. on each fiber.

Para-compact: Hausdorff + every open cover admits a part. of unity.

Compact Hausdorff, CW complex, metric space \( \Rightarrow \) para-compact

Prop. \( B \) para-compact \( \Rightarrow E \rightarrow B \) has an inner product.

\[ \Box \text{ Exercise.} \]

Step 2. Orthogonal complements

Prop. \( B \) para-compact, \( E_0 \rightarrow B \) subbundle of \( E \rightarrow B \).

\[ \exists E_0^\perp \text{ s.t. } E_0 \oplus E_0^\perp \cong E. \]

\[ \Box \text{ Choose inner product, } E_0^\perp \text{ = orthog. comp. in each fiber.} \]

Need to check local triviality

Over \( U \subseteq B \) choose \( m \) sections \( s_i \) for \( E_0 \), \( n-m \) for \( E \).

Apply Gram–Schmidt — continuous.

New sections trivialize \( E_0 \) & \( E_0^\perp \) simultaneously. \[ \Box \]

Note: \( E_0 \oplus E_0^\perp \cong E \)

via FACT above.
To prove that any $E$ has $E'$ with $E \oplus E'$ trivial, it now suffices to show:

**Prop.** $\mathcal{B} = \text{compact Hausdorff}$

Any $\mathbb{R}^n$-bundle $E \to \mathcal{B}$ is a subbundle of $\mathcal{B} \times \mathbb{R}^N$.

**Pf.** Choose: $U_i, \ldots, U_k$ s.t. $p^{-1}(U)$ trivial

$\; h_i : U_i \to U_i \times \mathbb{R}^n \to \mathbb{R}^n$

$q_i = \text{part of unity subord to } U_i$

Define: $g_i : E \to \mathbb{R}^n$

$v \mapsto (q_i(p(v)) h_i(v))$

linear inj. on each fiber with $q_i \neq 0$.

$g : E \to \mathbb{R}^{nk}$

$v \mapsto (g_1(v), \ldots, g_k(v))$

linear inj. on all fibers.

$f : E \to \mathcal{B} \times \mathbb{R}^{nk}$

$v \mapsto (p(v), g(v))$.

$\text{Im}(f)$ is a subbundle. Project in 2nd coord to get local triv. over $U_i$. 

\[ \square \]
THE GRASSMANN MANIFOLD.

We just showed

\[ [B, G_n] \longrightarrow \{ \mathbb{R}^n \text{- bundles over } B \} \]

is well defined. \( f \mapsto f^*(E_n) \)

Want to show it is a bijection. First, let's discuss the topology of \( G_n \) & \( E_n \).

\( G_n = \) set of all \( n \)-dim subspaces of \( \mathbb{R}^\infty \).
\( V_n = \) Stiefel manifold
\( = \) space of orthonormal \( n \)-frames in \( \mathbb{R}^\infty \).

\( V_n \) has a natural topology as a subspace of \( S^\infty \), and there is a quotient

\[ V_n \twoheadrightarrow G_n . \]

Endow \( G_n \) with quotient topology.

Define \( E_n = \{(l, v) \in G_n \times \mathbb{R}^\infty : v \in l \} \), \( p(l, v) = l \).

\underline{Lemma.} \( E_n \xrightarrow{p} G_n \) is a vector bundle.

\underline{Pf.} Let \( l \in G_n \), \( \pi_l : \mathbb{R}^\infty \rightarrow l \) orthog. proj.

\[ U_l = \{ l' \in G_n : \pi_l (l') \text{ has dim } n \} . \]

Steps: ① \( U_l \) open (check preim in \( V_n \) open).
② \( h : p^{-1}(U_l) \rightarrow U_l \times l \) is a local triv. \( (l', v) \mapsto (l', \pi_l(v)) \)

\( h \) clearly a bij, lin. iso on each fiber.

Need: \( h \), \( h^{-1} \) continuous (in alg).
THEOREM. \( X \) paracompact. The map \([X, \Gamma^n] \rightarrow \text{Vect}^n(X), \ f \mapsto f^*(\Gamma^n)\) is a bijection.

Example. \( M \subseteq \mathbb{R}^n \) submanifold. Define \( f: M \rightarrow \Gamma^n \) by \( x \mapsto T_x M \). Then \( TM \cong f^*(\Gamma^n) \).

Proof. Key observation: For \( E \rightarrow X \) an \( \mathbb{R}^n \)-bundle, an iso \( E \cong f^*(\Gamma^n) \) is equivalent to a map \( E \rightarrow \mathbb{R}^\infty \) that is a lin inj. on each fiber.

Indeed, given \( f: X \rightarrow \Gamma^n \) and \( E \xrightarrow{\xi} f^*(\Gamma^n) \) have:

\[
\begin{array}{cccc}
E & \xrightarrow{\xi} & f^*(\Gamma^n) & \rightarrow E_n & \rightarrow \mathbb{R}^\infty \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{f} & \Gamma^n
\end{array}
\]

Top row is the desired map.

Conversely, given \( g: E \rightarrow \mathbb{R}^\infty \) lin inj. on each fiber, define \( f: X \rightarrow \Gamma^n \) by \( x \mapsto g(p^{-1}(x)) \).

\( \tilde{f}: E \rightarrow E_n \) by \( v \mapsto g(v) \).

This gives diagram as above., by univ. prop. of pullbacks.
Surjectivity. Let $p : E \rightarrow X$ be an $\mathbb{R}^n$-bundle. (for simplicity, $X = \text{compact Hausdorff}$)

Choose cover $U_1, \ldots, U_n$ s.t. $E$ trivial over $U_i$.
& partition of unity $\varphi_1, \ldots, \varphi_n$.
Define $g_i : p^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
& $g : E \rightarrow \mathbb{R}^n \times \cdots \times \mathbb{R}^n \subseteq \mathbb{R}^\infty$
\[ v \mapsto (\varphi_1(g_1(v)), \ldots, \varphi_n(g_n(v))) \]
\[ \varphi_i \text{ means } \varphi_i \circ p = \text{scalar} \]
Check $g$ a lin. inj. on each fiber.

Injectivity. Say $E \cong f_0^*(\mathbb{R}^n), f_1^*(\mathbb{R}^n)$
for $f_0, f_1 : X \rightarrow G_n$.
\[ \Rightarrow g_0, g_1 : E \rightarrow \mathbb{R}^\infty \text{ lin inj on each fiber.} \]
To show $g_0 \sim g_1$ via maps that are lin inj on each fiber:
\[ \Rightarrow f_0 \sim f_1 \text{ via } f_t(x) = g_t(p^{-1}(x)). \]
Use:
\[ g_0 \quad \downarrow \quad g_1 \]
\[ \text{odd coords} \quad \xrightarrow{\ 1 \ } \quad \text{even coords} \]
\[ \text{straight line} \quad \xrightarrow{\ 3 \ } \quad \text{even coords} \]
\[ \text{N.B. } \ 3 \text{ only makes sense b/c } g_0, g_1 \]
\[ \text{are both maps from a fixed space } E \text{ to } \mathbb{R}^\infty. \]

\[ \text{e.g. } g_0 \xrightarrow{\ 1 \ } \text{odd coords via } (x_1, x_2, \ldots) \mapsto (1-t)(x_1, x_2, \ldots) + t(x_1, 0, x_2, \ldots) \]
At each stage, lin. inj. on fibers.

The Thm has an immediate corollary: v.b.'s over paracompact bases have inner products. Pull back obvious one on $\mathbb{R}^\infty$. \[ \square \]
We now know \([B, G_\eta] \leftrightarrow \{\text{vector bundles over } B\}\)
so \(\text{char. classes } \leftrightarrow H^*(G_\eta)\)

**Cell Structure on** \(G_\eta\).

First recall cell structure on \(G_1 = TRP^\infty\)
one \(i\)-cell \(e_i \quad \forall \ i\).
\(e_i\) glued to \(e_{i-1}\) by degree 2 map
\(e_i \leftrightarrow \{ \lambda \in TRP^\infty : \lambda \leq TR^{i+1} \}\)

Will generalize this.

A Schubert symbol \(\sigma = (\sigma_1, \ldots, \sigma_n)\) is a seq of integers
s.t. \(1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n\)

Let \(e(\sigma) = \{ \lambda \in G_\eta : \dim(\lambda \cap TR^{\sigma_i}) - \dim(\lambda \cap TR^{\sigma_{i-1}}) = 1 \quad \forall \ i \}\)

**Prop.** The \(e(\sigma)\) are the cells of a CW structure on \(G_\eta\).
\(\dim e(\sigma) = \sum_{i=1}^n (\sigma_i - i)\)

**Examples.** Consider in \(G_2\):
\(e(1,2) = \cdot\)
\(e(1,3) = \longrightarrow\)
\(e(2,3) = \square\)
Proof of Prop. Let $H_i = \text{hemisphere in } S^{n-1} \subseteq \mathbb{R}^n$

s.t. $\forall i$-coord non-neg.

\[ e(\sigma) \iff \{ (b_1, \ldots, b_n) \in V_n : b_i \in \text{int } H_i \} \]

Let $E(\sigma) = \{ (b_1, \ldots, b_n) \in V_n : b_i \in H_i \}$

Main step: $E(\sigma)$ a closed ball of dim $\Sigma(n-1)$

$n=1$ case: $E(\sigma) = H_1 \checkmark$

$n > 1$ case: Define \( \pi : E(\sigma) \to H_1 \)

\[(b_1, \ldots, b_n) \mapsto b_1 \]

\[ p : E(\sigma) \to \pi^{-1}(e_{\sigma_1}) \]

rotate fiber over $b_1$ to $\pi^{-1}(e_{\sigma_1})$

by rotating $b_1$ to $e_{\sigma_1}$, fixing orthog. comp. of $\langle b_1, e_{\sigma_1} \rangle$

Then $\pi \times p : E(\sigma) \to H_1 \times \pi^{-1}(e_{\sigma_1})$

is a contin. bij $\Rightarrow$ homeo.

(exercise: Hausdorff)

Remains to check $\pi^{-1}(e_{\sigma_1})$ a ball.

Induct on $n$. $\pi^{-1}(e_{\sigma_1}) \iff E(v_{n-1}, \ldots, v_{n-1})$

Span takes int $E(\sigma)$ to $e(\sigma)$ bijectively.

Since $G_n$ has quotient top. from $V_n \to$ homeo.

Need to check that the CW complex obtained from the $E(\sigma)$ give right topology. Induct on skeleta. \(\square\)
Other versions: \( \text{Vect}_c^0 (X) \leftrightarrow [X, \text{Gn}(C)] \)
\( \text{Vect}_c^0 (X) \leftrightarrow [X, \tilde{\text{Gn}}] \)

Note \( \text{Vect}_c^0 (S^1) \) trivial \( \Rightarrow [S^1, \tilde{\text{Gn}}] \) trivial
\( \Rightarrow \pi_1 (\tilde{\text{Gn}}) = 1. \)
\( \Rightarrow \tilde{\text{Gn}} = \text{univ. cover of Gn.} \)

For \( f: X \rightarrow \text{Gn} \), \( f^*(E) \) orientable iff
\( f \) lifts to \( \tilde{\text{Gn}} \) & in this case, orientations correspond to choices of lifts.

Prop. \( \text{Gn} \) is a manifold.

Pf. Clear for interior of a top-dim. cell.
But \( \text{Gn} \) is homogeneous: \( \exists \) homeo taking any pt to any other pt, ie the one induced by a linear map.
Stiefel-Whitney and Chern Classes

First, we will show that characteristic classes exist by defining specific ones, the SW classes $W_i$ and the Chern classes $c_i$. Then we will show these are all char. classes (in the $\mathbb{R}, \mathbb{Z}_2$ & $\mathbb{C}, \mathbb{Z}$ cases, resp.) by computing $H^*(G_n; \mathbb{Z}_2)$ and $H^*(G_n(\mathbb{C}); \mathbb{Z})$.

**Thm.** Exist seq. of fns $W_1, W_2, \ldots$ assigning to each real
V.b. $E \to B$ a class $W_i(E) \in H^i(B; \mathbb{Z}_2)$ s.t.
(i) $W_i(f^*(E)) = f^*(W_i(E))$
(ii) $W(E_1 \oplus E_2) = W(E_1) \cup W(E_2)$, $w = 1 + W_1 + W_2 + \ldots$
(iii) $W_i(E) = 0$ if $i > \dim E$
(iv) $W_i$ (canon. bundle $\to \mathbb{R}P^\infty$) is gen. of $H^i(\mathbb{R}P^\infty; \mathbb{Z}_2)$.

$W =$ total SW class. (iii) $\Rightarrow$ it is a finite sum.
(ii) is Whitney sum formula.
(iv) $\Rightarrow$ the $W_i$ are not all zero!
(i) $\Rightarrow W_i(B \times \mathbb{R}^n) = 0$ if $i > 0$. (ii) $\Rightarrow$ $W_i$ stable. \underline{Cor:} $W_i(TS^n) = 0$ if $i > 0$.

For complex bundles, have $c_i \in H^{2i}(B; \mathbb{Z})$. Thm is same except:
(iv) $c_i$ (canon. $\to \mathbb{C}P^\infty$) gen. $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

Proof requires one tool from alg. top ...
The Leray-Hirsch Theorem

When does $H^*(E)$ look like $H^*(F \times B)$?  First, recall:

Künne\text{\textsc{h}} Formula. $H^*(X; R) \otimes_R H^*(Y; R) \cong H^*(X \times Y; R)$

\[ a \otimes b \mapsto p_!^*(a) \cup p_2^*(b) \]

For a fiber bundle, $H^*(E) \to H^*(F)$ not nec. surj., so don't always have a map the other way. To get a Künne\text{\textsc{h}}-like formula, must add this to the assumptions.

General theme in bundle theory: try to extend an object related to the fiber (inner prod, cohom. class) to whole bundle.

L-H Theorem. Let $F \to E \to B$ be a fiber bundle, $R$ a ring st.

(i) $H^*(F; R)$ is a free f.g. $R$-module for all $n$.

(ii) \exists $c_j \in H^{k_j}(E; R)$ s.t. the $i^*(c_j)$ form a basis for $H^*(F; R)$

Then:

\[ H^*(B; R) \otimes_R H^*(F; R) \cong H^*(E; R) \]

\[ \sum b_i \otimes i^*(c_j) \mapsto p^*(b_i) \cup c_j \]

In other words: $H^*(E; R)$ a free $H^*(B; R)$ module w/basis $c_j$.

Module structure given by $\cup$.

- The $c_j$ do exist for product bundles: pull back via projection.
- The $c_j$ do not exist for $S^1 \to S^3 \to S^2$ as $H^i(S^3) = 1$. 
Pf. of \textbf{LH} (a few words) Using long ex. seq. for a pair, plus excision, you reduce to understanding
\[ p^{-1}(B^{n-1}) \to B^{n-1} \quad \text{(n-skeleton)} \]
\[ p^{-1}(n-cell) \to n-cell \]
Former works by induction, latter by local triviality. \[\blacksquare\]

\underline{Pf. of SW Thm.} \quad \eta: E \to B

\[ \eta: \eta(E) \to B \quad \eta(E) = \text{space of lines} \]
\[ \text{fibers } \text{TRP}^{n-1} \]

To use \textbf{L-H}, need \( x_i \in H^i(\eta(E); \mathbb{Z}_2) \)
restricting to gens for \( H^i(\text{TRP}^{n-1}; \mathbb{Z}_2) \).
\( (E \to B) \to q: E \to \text{TR}^\infty \text{ lin. inj on fibers} \)
\[ \eta: P(q): P(E) \to \text{TR}^\infty \]

Let \( \chi = \text{gen} \text{ for } H^i(\text{TRP}^\infty; \mathbb{Z}_2) \)
\[ \chi = P(q)^*(\chi) \quad \leftarrow \text{easy to see this generates } H^i(\text{fiber}). \]
\[ x_i = \chi^i. \quad \text{also indep. of } g \]

\[ \text{L-H } \Rightarrow H^*(\eta(E)) \text{ a free } H^*(B)-\text{module with} \]
\[ \text{basis } 1, x, \ldots, x^{n-1} \]
\[ \Rightarrow x^n = \text{unique linear combo:} \]
\[ x^n + W_1(E)x^{n-1} + \cdots + W_n(E). 1 = 0 \]
for some \( W_i(E) \in H^i(B; \mathbb{Z}_2) \).

Also set \( W_i(E) = 0 \text{ for } i > n \)
\[ W_0(E) = 1. \]

These are the SW classes. Need to check properties (i)-(iv), uniqueness.
(i) Naturality

\[ E' \xrightarrow{\tilde{f}} E \xrightarrow{g} \mathcal{R}^\infty \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ B' \xrightarrow{f} B \]

\[ \implies P(\tilde{f})^* x(E) = x(E') \]
\[ \implies P(\tilde{f})^* x_i(E) = x_i(E') \]

Commutativity \Rightarrow \text{module structure pulls back}

i.e., \[ x^n + w_i(E)x^{n-1} + \cdots + w_n(E) \cdot 1 = 0 \]
\[ \implies x^n + f^*(w_i(E)) x^{n-1} + \cdots + f^*(w_n(E)) \cdot 1 = 0 \]

But this defines \( w_i(E') \) so \( w_i(E'') = f^*(w_i(E)) \) \( \forall i' \).

(ii) Whitney sum - similar flavor

(iii) \( w_i(E) = 0 \) \( i > n \) by definition.

(iv) \( W_1(CB \to \mathcal{R}P^\infty) \neq 0 \).

Almost by definition: \( x(\text{loop in } P(E)) \) measures whether or not a line comes back to where it started with same or different orientation.

\[ x + W_1(CB) \cdot 1 = 0. \]
\[ \implies W_1(CB) = x. \]
For uniqueness of $w_i$, need a tool.

Splitting Principle. Given $E \to B \ni f: A \to B$ s.t.

(i) $f^*(E)$ splits as a sum of line bundles
(ii) $f^*: H^*(B) \to H^*(A)$ injective

Now, the $w_i$ are unique because:

(iv) determines $w_i(CB \to \mathbb{RP}^\infty)$
(iii) determines $w_i(CB \to \mathbb{RP}^\infty)$ $i > 1$.
(i) determines $w_i$ (line bundles)
(ii) determines $w_i$ (sum of line bundles)

SP + (i) determines $w_i$ (any bundle).

Pf of SP. $A = F(E) =$ flag bundle of $E$

= space of orthog. splittings $l_1 \oplus \cdots \oplus l_n$

of $E$ into lines

$f: A \to B$ projection

$f^*(E) = \{(\text{splitting of fiber over } b, \text{ vector in fiber over } b)\}$

This has $n$ obvious linear subbundles, which give
the splitting.

For (ii) use Leray-Hirsch $\Rightarrow H^*(B): 1$ a summand of $H^*(A)$.
**Important Example.**

\[(E_i)^n \to (G_i)^n \quad \text{ } E_i = \text{Canon. line bundle} \]

\[(E_i)^n \cong \bigoplus \pi_i^*(E_i) \quad \pi_i : (G_i)^n \to G_i \quad \text{true for any } E^n \to B^n \]

\[\Rightarrow w((E_i)^n) = \prod (1+\alpha_i) \in \mathbb{Z}_2[\alpha_1, \ldots, \alpha_n] \cong H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \]

\[\Rightarrow w_i((E_i)^n) = i^{th} \text{ symm. poly } \pi_i \text{ in the } \alpha_j \]

e.g. for \(n=3\):  
\[
\begin{align*}
\pi_1 &= \alpha_1 + \alpha_2 + \alpha_3 \\
\pi_2 &= \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 \\
\pi_3 &= \alpha_1 \alpha_2 \alpha_3
\end{align*}
\]

So all \(w_i \) non-zero \(i \leq n\).

**Next:** We'll use this to show

\[\mathbb{Z}_2[\omega_1, \ldots, \omega_n] \to H^*(G_n; \mathbb{Z}_2)\]
COHOMOLOGY OF GRASSMANIANS

We showed \( w_i ((E_i)^n \to (G_1)^n) \neq 0 \) \( 0 \leq i \leq n \).
Naturality \( \Rightarrow \) \( w_i (E_n) \neq 0 \) \( 0 \leq i \leq n \).

Let \( f: (\mathbb{RP}^n)^n \to G_n \) be classifying map for \((E_1)^n\).
& \( W_i = w_i (E_n) \).
Then:
\[
\mathbb{Z}_2 [w_1, \ldots, w_n] \to H^* (G_n; \mathbb{Z}_2) \xrightarrow{f^*} H^* (\mathbb{RP}^n)^n; \mathbb{Z}_2) \cong \mathbb{Z}_2 [\alpha_1, \ldots, \alpha_n]
\]

sends \( w_i \) to \( i^{th} \) symm. poly. \( \alpha_i \) in the \( \alpha_j \).

Fact. The \( \alpha_i \) are alg. indep.

\( \Rightarrow \) above map is inj
\( \Rightarrow \) \( \mathbb{Z}_2 [w_1, \ldots, w_n] \hookrightarrow H^* (G_n; \mathbb{Z}_2) \).

Thm \( H^* (G_n; \mathbb{Z}_2) = \mathbb{Z}_2 [w_1, \ldots, w_n] \)
also: \( H^* (G_n (G); \mathbb{Z}) = \mathbb{Z} [c_1, \ldots, c_n] \)

\( \text{Pf.} \) We showed \( \text{im} f^* \) contains \( \mathbb{Z}_2 [\alpha_i, \ldots, \alpha_n] \)
Also \( \text{im} f^* \) contained in \( \mathbb{Z}_2 [\alpha_i, \ldots, \alpha_n] \) since permuting the \( \mathbb{RP}^n \) factors gives same bundle with \( \alpha_i \)'s permuted.
So:
\[
\mathbb{Z}_2 [w_1, \ldots, w_n] \to H^* (G_n; \mathbb{Z}_2) \xrightarrow{f^*} \mathbb{Z}_2 [\alpha_i, \ldots, \alpha_n] \cong \mathbb{Z}_2 [\alpha_1, \ldots, \alpha_n]
\]
\( f^* \) surjective. To show
\( f^* \) injective.
Focus on $r$-grading:

$$(\mathbb{Z}_2[\omega_1, \ldots, \omega_n])_r \rightarrow \mathcal{H}_r^r(G_n; \mathbb{Z}_2) \rightarrow (\mathbb{Z}_2[\omega_1, \ldots, \omega_n])_r$$

Since composition surj, suffices to show $\dim \mathcal{H}_r^r(G_n; \mathbb{Z}_2) 
\leq \dim (\mathbb{Z}_2[\omega_1, \ldots, \omega_n])_r$.

Let $p(r, n) = \# \text{partitions of } r \text{ into } n \text{ nonneg integers}$.

Step 1. $\dim (\mathbb{Z}_2[\omega_1, \ldots, \omega_n])_r = p(r, n)$.

$w_1^r w_2^{r_2} \cdots w_n^{r_n} \in (\mathbb{Z}_2[\omega_1, \ldots, \omega_n])_r$ means

$r_1 + 2r_2 + \cdots + nr_n = r$ (since $w_i \in \mathcal{H}_i^i$)

$\rightarrow$ partition of $r$: $r_n \leq r_{n-1} \leq \cdots \leq r_1$

Step 2. $\dim \mathcal{H}_r^r(G_n; \mathbb{Z}_2) \leq \# \text{ Schubert cells of dim } r$.

General fact about cell complexes

Step 3. $\# \text{ Schubert cells in } G_n \text{ of } \dim r = p(r, n)$.

A partition $a_1 \leq a_2 \leq \cdots \leq a_n$

$\rightarrow$ Schubert symbol $(a_1+1, a_2+2, \ldots, a_n+n)$.

Example. $r = 10$, $n = 6$.

partition: 0, 0, 1, 1, 3, 5
Schubert cell: (1, 2, 4, 5, 8, 11)
monic: $w_1^2 w_2^4 w_4$
The Group of Line Bundles

We'll first show: \( \text{Vect}^{1}(X) \) is a group under \( \otimes \).
and then: \( \text{Vect}^{1}(X) \cong H^{1}(X; \mathbb{Z}) \). The isom. is \( \omega \).

Gluing construction of vector bundles. Given \( p: E \to B \), \( \{ U_{\alpha} \} \),
\( h_{\alpha}: p^{-1}(U_{\alpha}) \to U_{\alpha} \times \mathbb{R}^{n} \), can recover
\( E = ( \bigcup U_{\alpha} \times \mathbb{R}^{n} ) / \sim \)
where \((x,v) \in U_{\alpha} \times \mathbb{R}^{n} \equiv h_{\alpha}^{-1}(x,v) \in U_{\beta} \times \mathbb{R}^{n} \quad x \in U_{\alpha} \cap U_{\beta} \).
Write \( g_{\alpha x} \) for the gluing func. \( h_{\alpha}^{-1}: U_{\alpha} \cap U_{\beta} \to GL_{n}(\mathbb{R}) \).
\( g_{\alpha x} \) cocycle condition: \( g_{\alpha x} g_{\beta x} = g_{\gamma x} \) on \( U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \).
Conversely: any collection of gluing functions satisfying

cocycle cond gives rise to a vector bundle.

The gluing functions for \( E_{1} \otimes E_{2} \) are the tensor products of
the gluing functions for \( E_{1}, E_{2} \).

In general, \( \otimes \) on \( \text{Vect}^{1}(X) \) is comm, assoc, and has
identity = trivial line bundle.

For \( n = 1 \), also have inverses. In fact, each elt is its
own inverse.

Example. Möbius \( \to S^{1} \) has gluing fns 1, -1
\( 1 \otimes 1 = 1 \quad -1 \otimes -1 = 1 \)
\( \Rightarrow \) Möbius \( \otimes \) Möbius \( \to S^{1} \) is trivial.
For general line bundles, we obtain inverse by replacing gluing matrices by their inverses, as \( t \circ t^{-1} = 1 \).

Cocycle condition still works since 1x1 matrices commute.

Endow \( E \) w/inner product \( \Rightarrow \) rescale all \( h_x \) with isometries

\( \Rightarrow \) all gluing \( f_x \) \( \pm 1 \).

\( \Rightarrow \) gluing \( f_x \) for \( E \otimes E \) all 1.

\( \Rightarrow \) \( E \otimes E \) trivial.

We have: \( \text{Vect}'(X) = [X, G_1] \cong H^1(X; \mathbb{Z}_2) \)

\[ \begin{array}{c}
\text{isom. of} \\
\text{sets}
\end{array} \quad \begin{array}{c}
\uparrow \\
\uparrow \text{since } G_1 = \operatorname{RP}^\infty \text{ is } K(\mathbb{Z}_2, 1).
\end{array} \]

Prop. \( W_1 : \text{Vect}'(X) \cong H^1(X; \mathbb{Z}_2) \quad X = \text{CW-complex} \).

\[ \begin{array}{c}
\text{If.} \\
\text{First show } W_1 \text{ a homomorphism.}
\end{array} \]

Step 1. \( W_1(L_1 \otimes L_2) = W_1(L_1) + W_1(L_2) \)

for \( L_i \rightarrow G_1 \times G_1 \) the pullback of \( E_1 \rightarrow G_1 \)

via \( \pi_i : G_1 \times G_1 \rightarrow G_1 \).

Have \( H^*(G_1 \times G_1) \cong \mathbb{Z}_2[x_1, x_2] \otimes \mathbb{Z}_2[x_1, x_2] \cong \mathbb{Z}_2[x_1, x_2] \)

\( H^*(G_1 \times G_1) \cong \mathbb{Z}_2[x_1, x_2] \)

This is an isom. on \( H^1 : \mathbb{Z}_2 \otimes \mathbb{Z}_2 = \{0, x_1, x_2, x_1 + x_2\} \)

So suffices to compute \( W_1(L_1 \otimes L_2 \rightarrow G_1 \times G_1) \)

Over \( G_1 \times \mathbb{Z}^* \), \( L_2 \) trivial \( \Rightarrow L_1 \otimes L_2 \cong L_1 \otimes 1 \cong L_1 \)

Similar for \( \mathbb{Z} \times G_1 \)

\( \Rightarrow \) \( W_1(L_1 \otimes L_2) = \alpha_1 + \alpha_2 = W_1(L_1) + W_1(L_2) \).

\( \uparrow \) Use naturality of pullback via \( G_1 \rightarrow G_1 \times G_1 \).
Step 2. (Naturality) $E_1, E_2$ arbitrary bundles
$E_i = f_i^*(E_i)$ $f_i : X \to G_i$

Let $F = (f_1, f_2) : X \to G_1 \times G_1$
$F^*(L_i) = f_i^*(E_i) = E_i$

follow your nose...

$W_i(E_1 \otimes E_2) = W_i(F^*(L_1) \otimes F^*(L_2)) = W_i(F^*(L_1 \otimes L_2))$
\[= F^*(W_i(L_1 \otimes L_2)) = F^*(W_i(L_1) + W_i(L_2))\]
\[= F^*(W_i(L_1)) + F^*(W_i(L_2))\]
\[= W_i(F^*(L_1)) + W_i(F^*(L_2))\]
\[= W_i(E_1) + W_i(E_2).\]

The isomorphism $[X, G_i] \longrightarrow H^i(X; \mathbb{Z}_2)$

is $[f] \longmapsto f^*(x)$

It factors as $[X, G_i] \longrightarrow \text{Vect}^i(X) \longrightarrow H^i(X; \mathbb{Z}_2)$

$[f] \longmapsto f^*(E_i) \longmapsto W_i(f^*(E_i)) = f^*(W_i(E_i)) = f^*(x)$

First map is bij, comp is isom $\Rightarrow$ 2nd map bij. □

We can unravel the last step. Want to define

$H^i(X; \mathbb{Z}_2) \longrightarrow \text{Vect}^i(X)$

inverse to $W_i$. Given $q \in H^1$, define an $\mathbb{R}$-bundle skeleton by skeleton. On 1-skeleton, use $q$ to decide between Moebius & trivial bundle. As $q$ is a cocycle, it is trivial on any loop bounding a 2-cell, so can extend over 2-skeleton and higher.
THE EULER CLASS

\[ e \in H^n(\mathbb{C}^n; \mathbb{Z}) \]

\( \sim e \) is \( n \)-dim class for oriented \( \mathbb{R}^n \)-bundles

idea: given \( n \)-chain, put it in gen. pos. wrt 0-section, count intersection points with sign.

The Euler class satisfies:

1. \( e(f^*(E)) = f^* e(E) \)
2. \( e(-E) = - e(E) \)
3. \( e(E_1 \oplus E_2) = e(E_1) \cup e(E_2) \)
4. \( e(E) = - e(E) \) if \( n \) odd (i.e. \( e(E) \) is 2-torsion)
5. \( e(E) = 0 \) if \( E \) has nonzero section
6. \( \langle e(M), [M] \rangle = \chi(M) \)

Instability: Unlike \( wi, ci \), the class \( e \) is unstable:

\[ e(E \oplus \text{trivial}) = 0 \] (nonvanishing section)

The construction of \( e \) requires one tool.

Let \( E' = E - 0 \text{-sec.} \)

We'll show \( \exists C \in H^n(E, E') \) restricting in each fiber to a gen for \( H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \).

\( C = \text{Thom class} \)

Define \( e = \text{restriction of } C \) to 0-section: \( H^*(E, E') \to H^*(E) \to H^*(B) \)

This does just what we want:

\[ \text{To compute, perturb intersections to lie in fibers.} \]
Thom Isomorphism

Orientability. \( \mathbb{R}^n \to E \to B \rightsquigarrow \) disk bundle, \( D^n \to D(E) \to B \) and sphere bundle \( S^{n-1} \to s(E) \to B \)

Say \( E, D(E) \) orientable if \( s(E) \) is \( S^1 \) orientable if the map \( H^{n-1}(S^{n-1}; \mathbb{Z}) \) induced by any loop in \( B \) is \( \text{id} \).

E.g. \( T^n \) is orientable \( S^1 \) bundle over \( S^1 \), K.B. nonorientable.

Thom class. A Thom class is a \( c \in H^n(D(E), s(E); \mathbb{Z}) \) restricting to \( \gamma_0 \) for \( H^n(D^0, S^{n-1}; \mathbb{Z}) \) in each fiber.

Thm. \( E \) orientable \( \Rightarrow \) \( c \) exists.

Thom isomorphism. The map \( H^i(B; \mathbb{Z}) \to H^{i+n}(D(E), s(E); \mathbb{Z}) \)

\[ b \mapsto p^*(b) \cup c \]

is isom. \( \forall i \geq 0 \), and \( H^i(D(E), s(E); \mathbb{Z}) = 0 \) \( i < n \).

Thom space. \( T(E) = D(E)/s(E) \) disk fibers \( \rightsquigarrow \) spheres in \( T(E) \), all spheres meet at basept.

Thom class \( \leftrightarrow \) elt of \( H^n(T(E); \mathbb{Z}) \nsimeq H^n(T(E); \mathbb{Z}) \).

Restricting to gen. of \( H^0(S^n; \mathbb{Z}) \) in each "fiber" Thom isom \( \to H^i(B; \mathbb{Z}) \nsimeq H^{i+n}(T(E); \mathbb{Z}) \)

\( T(E) \) central to Thom's work on cobordism.
**Thom Class**

* all coeffs = \( \mathbb{Z} \)

**Thm.** Every orientable bundle \( E \to B \) has a Thom class

**Pf.** Assume \( B \) is connected finite dim CW complex.

Claim. \( H^i(D(E), S(E)) \xrightarrow{\sim} H^i(D^0, S^{n-1}) \quad \forall \) fibers.

Say \( B \) is \( k \)-dim, assume true for smaller dim complexes.

For concreteness \( i = n \). Other cases easier.

Set \( U = \text{nbd of } B^{k-1}, V = \text{open } k \)-cells

**Mayer-Vietoris:**

\[
0 \to H^n(D(E), S(E)) \to H^n(D(E) \cup S(E) \cup S(E)) \xrightarrow{\psi \text{ diff map}} H^n(D(E) \cup S(E) \cup S(E))
\]

\[
\begin{array}{c}
H^n(D^0, S^{n-1}) \\
\oplus \\
H^n(D^0, S^{n-1})
\end{array}
\]

\[
\begin{array}{c}
\text{induction} \\
\text{middle consistently} \\
\text{for mod 2 version skip this step.}
\end{array}
\]

Can rewrite everything with \((E, E-\text{ (0-sect)})\) & \((R^n, R^n-0)\)

Relative LH \( \Rightarrow H^*(D(E), S(E)) = \text{free } H^*(B) \)-module with basis \( c \)

\( \cong H^*(B) \)

This is the Thom isomorphism.
**Properties of the Euler Class**

1. **Naturality.** A pullback $f^*(E)$ comes with a map $f^*(E) \xrightarrow{f} E$ that is a lin. isom. on fibers. Thus $\tilde{f}$ pulls back the Thom class to a Thom class: $\tilde{f}^*(c(E)) = c(f^*(E))$.

   $\tilde{f}|_{B} = f$ so when we pass through

   $H^*(E, E') \rightarrow H^*(E) \rightarrow H^*(B)$

   we get the result.

2. **Negation.** Basically obvious — negating the orientation of $E$ negates all signs of intersection.

3. **Whitney sum.** Consider $p_i : E_i \oplus E_{-1} \rightarrow E_i$. (linear on fibers)

   Say $c(E_i) \in H^n(E_i, E_{i-1})$ and $c(E_{-1}) \in H^n(E_{-1}, E_{-1})$.

   Want: $p_i^*(c(E_1)) \cup p_{-1}^*(c(E_{-1})) = c(E_1 \oplus E_{-1})$

   Reduces to showing

   $H^m(\mathbb{R}^{m+n}, \mathbb{R}^n) \times H^m(\mathbb{R}^{m+n}, \mathbb{R}^m) \rightarrow H^{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^n \setminus 0)$

   takes $(\text{gen}, \text{gen}) \rightarrow \text{gen}$.

4. **Odd dimensions.** Use (2) plus the fact that negation is an orientation reversing automorphism.

5. **Nonvanishing sections.** Basically obvious — in the presence of a nonvan. section, any $n$-chain in $B$ can be pushed completely off of $B$. 
(6) Euler characteristic

We know \( \langle e(M), M \rangle = \text{self-int of } M \text{ in } TM \)

Step 1. \( \langle e(M), M \rangle = \text{self-int of } \Delta \text{ in } M \times M \).
Step 2. Latter = sum of indices of Lefschetz fixed pts of an \( f: M \rightarrow M \).
Step 3. Choose an \( f \) and compute.

Step 1. Self-int of \( M \) in any \( 2n \)-dim man. \( U \) equals \( \langle e(NuM), M \rangle \).
Remains to show: \( N_{M \times M} \Delta \cong TM \).
A vector \( (u,v) \in T_x M \times T_x M \cong T_{(x,x)} M \times M \times M \)
is tangent to \( \Delta \iff u = v \)
hence normal to \( \Delta \iff u = -v \).
The isomorphism \( TM \rightarrow N_{M \times M} \Delta \) is
\[
(x,v) \mapsto ((x,x), (v,-v)).
\]

Step 2. \( f: M \rightarrow M \) is Lefschetz if \( Df-I \) invertible at each pt.
The index of \( f \) at a fixed pt is +1 if \( \det(Df-I) > 0 \), -1 o.w.
This number equals the sign of intersection of \( \Gamma(f) \) with graph \( \Gamma(f) \) at \( (x, f(x)) \).

Idea: Check sign of \( \Gamma(f) \) at \( (V_1, V_1), \ldots, (V_n, V_n), (V_1, Df(V_1)), \ldots, (V_n, Df(V_n)) \).

Claim follows.
But \( \Delta \cap \Gamma(f) = \Delta \cap \Delta \), so done.

\[
\begin{align*}
|I f| &= |I I| \\
&= |Df-I|
\end{align*}
\]
Step 3. Find a nice Lefschetz function.
Choose a vector field, say one $\mathbf{F}$ pointing from barycenters of higher dim. simplices to barycenters of lower dim. simplices (actually, gradient flow for any Morse fn will work).

At a vertex:

Then $f$ is time $\varepsilon$ flow.
In the 3 cases, $Df$ is $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
So $\det(Df-I)$ is $+$ $-$ $-$ $+$ as desired.
The Thom Isomorphism reduces to a rel version of Leray-Hirsch.

Fiber bundle pairs. $\bullet F \to E \to B$ with $E' \subseteq E$ s.t. $E' \to B$ a bundle with fibers $F' \subseteq F$, compatible trivializations $\sim (E, E') \to B$ e.g. $S(E) \subseteq D(E)$

Thm (Relative Leray-Hirsch). Say $(F, F') \to (E, E') \to B$ a f.b. pair s.t. $H^*(F, F')$ f.g. free $R$-mod in each dim. If $3 \ c_j \in H^*(E, E')$ whose restrictions form a basis for $H^*(F, F')$ in each fiber then $H^*(E, E')$ is free $H^*(B)$-module w/ basis $\{c_j\}$.

Idea: Construct a related bundle $\hat{E}$, apply absolute to $\hat{E}$.

Construction of $\hat{E}$. Let $M = \text{mapping cyl of } p: E' \to B$ note $E' \subseteq M$

$\hat{E} = M \amalg_{E'} E$

$\hat{F} = \text{cone on } \hat{E} = \text{mapping cyl of const. map}$

Key isomorphism. $H^* (\hat{E}) \cong H^*(\hat{E}, B) \oplus H^*(B)$ as $H^*(B)$ modules $\vec{w}$

$H^*(E, E') \leftrightarrow \text{killing } E' \text{ in } E \text{ same as killing } M \text{ in } \hat{E}, \text{ same as killing } B \text{ in } \hat{M} \text{ in } \hat{E}$. $\ast$ splitting from retraction $p: \hat{E} \to B$.

Let $\hat{c}_j$ correspond to $(c_j, 0)$. The $c_j \& 1$ restrict to basis for $H^*(\hat{F}) \cong H^*(F, \hat{F}')$

$LH \Rightarrow H^*(\hat{E})$ free $H^*(B)$-modules, basis $\{1, \hat{c}_j\}$

$\Rightarrow c_j \text{ free basis for } H^*(E, E')$. \[ \square \]
Euler Class via Poincaré Duality

Fix some oriented $\mathbb{R}^n \to E \to B$ = smooth, oriented, $k$-manifold.
Let $D = \text{disk bundle of } E$.
$D$ is an $(n+k)$-manifold with $d$, so it has Poincaré duality
\[
H^i(M, \partial M) \cong H^{n-k-i}(M)
\]
\[
\alpha \mapsto [M] \cap \alpha = \alpha^*
\]
relative fundamental class

Regard the fundamental class $[B]$ as elt of $H_k(D)$
via the map on $H_*$ induced by $B \hookrightarrow D$.

Prop. $[B] = c^*$ in $H_k(D)$.
Thom class

So: An explicit cochain $\{2\text{-cells of } B\} \to \mathbb{Z}$ representing $u$
is given by counting intersections of a section with 2-cells of $B$
(assuming gen. pos.). Actually, can replace the section with
any subspace, homotopic/homologous to $B$.

Prf. Apply three isomorphisms (WLOG $B$ connected):
\[
\mathbb{Z} = H^0(B) \xrightarrow{\text{Thom}} H^0(D, S^1) \xrightarrow{\text{sphere bundle}} PD \xrightarrow{\text{PD}} H_k(D) \xrightarrow{\text{}} H_k(B) = \mathbb{Z}
\]
\[
H^0(D, \partial D)
\]
\[
1 \to C \to C^*
\]
Since the composition $\mathbb{Z} \to \mathbb{Z}$ is an iso, $C^* = \pm [B]$.
(Must work harder to get the sign.)
Circle Bundles and the Euler Class

There are correspondences:

\[ C^1 \text{-bundles} \leftrightarrow \text{oriented } \mathbb{R}^2 \text{-bundles} \leftrightarrow \text{oriented } S^1 \text{-bundles} \]

Both \( \rightarrow \) are easy.

First \( \leftarrow \) via Euclidean metric. \( C \)-structure is rotation by \( \pi \).

Second \( \leftarrow \) uses \( \text{Diff}^+(S^1) \circ \text{Isom}^+(S^1) \cong S^1 \).

This implies we can modify the local trivializations so they remember distance on \( S^1 \). Then build \( \mathbb{R}^2 \)-fibers by coning off \( S^1 \)-fibers.

Key example. (Hopf bundle \( S^1 \to S^3 \to S^2 \) \( \leftrightarrow \) \( \mathbb{C} \mathbb{L} \to \mathbb{C} P^1 \))

\( C \)-description: \( S^3 = \{(z,w) \in \mathbb{C}^2 : \# |z|^2 + |w|^2 = 1 \} \)

\( (z,w) \mapsto w/z \in \mathbb{C} \)

or \( (z,w) \mapsto \) line spanned by \( (z,w) \in \mathbb{C} P^1 \)

Topological description

There are two \( \mathbb{D}^3 \times S^1 \)

The bundles over the two \( \mathbb{D}^3 \) are equal as sets \( \sim \) a map \( S^3 \to S^2 \)
Euler class via sections of $S^1$-bundles

A bundle $S^1 \to E \to X$ is trivial iff it has a section. For $X = \text{CW complex}$, can try to build a section inductively over skeleta. Say $S_1$ is section over $X^{(i)}$. $S_1$ extends over $D^{i+1}$ iff $S_1 \simeq \partial D^{i+1} \hookrightarrow X^{(i)}$ $S_1 \to S^1$ is homot. trivial. But we know: $\pi_i(S^1) = \begin{cases} \mathbb{Z} & i = 1 \\ 0 & \text{otherwise} \end{cases}$ (exercise)

So only obstruction is over 2-skeleton.

Can use this idea to build a cochain $\{2\text{-cells of } X \} \to \mathbb{Z}$.

Step 1. Choose any section $S_1$ over $X^{(1)}$.

Step 2. Take degrees of maps $\partial D^2 \to S^1$ as above.

Can check directly this is a cocycle. It vanishes $\Rightarrow$ trivial bundle.

(see Cordei-Conlon).

It turns out this is the Euler class. See below.

We will show:

$e_1$ for $C^1$-bundles $\leftrightarrow$ $e$ for or. $\mathbb{R}^2$-bundles $\leftrightarrow$ $e$ for or. $S^1$-bundles.

We already showed: $G_1 \text{ Vect}_C(X) \overset{\cong}{\longrightarrow} H^2(X; \mathbb{Z})$

For $X = \Sigma g$ can build explicitly $E_k$ s.t. $e(E_k) = k \in \mathbb{Z} \cong H^2(\Sigma g; \mathbb{Z})$.

Idea: Remove a 2-cell. Take trivial bundle over complement, trivial over 2-cell, glue with a twist on $\partial = T^2$

Dehn surgery on $\Sigma g \times S^1$

Dehn twist in fiber direction.
Exercise. \( g = 0 \quad E_k = L(k, 1) \quad L(0, 1) = S^2 \times S^1 \)

\[
\text{note } L(2, 1) = UT_{S^2} \text{ since have same Euler class.}
\]

\( g = 1 \quad E_k = M[k_1] \quad \text{II}_{1\mathbb{R}P^3} \)

Prop. For \( C \to E \to X, \quad c_1 = e = e. \)

Pf. First compare \( e \) for \( S^1 \)-bundles with \( C_1 \).
If we believe \( e \) is a char class, then we know it is a deg 1 poly
in the \( c_1 \) \( \implies \) it is a multiple of \( C_1 \).

So suffices to check on \( CLB \to CP^1 \).

By defn \( C_1(CLB) = \kappa = 1 \in \mathbb{Z} \in H^2(CP^1) \).

We choose trivializations of the circle bundle \( S^1 \to S^3 \to S^2 \)
over \( \Delta, \Delta^c \) and show corresponding sections over \( S^1 = \partial \Delta \)
intersect in one pt. This means (up to sign) \( e = 1 \).

Over \( \Delta \): \( \kappa \mapsto (\kappa, 1)/\text{norm} \)

\( \Delta^c \): \( \kappa \mapsto (1, \kappa)/\text{norm} \quad (\infty \mapsto 0) \)

On \( \partial \Delta \) these equal only for \( \kappa = 1 \).
exercise: check \( e \) for top. description.

We'll also show the two \( e \)'s have same. for \( X \) a manifold.
Idea: Suppose have a section of \( E \) over \( \partial \mathbb{R}^2 \) of degree 1.

i.e. \( (1, \Theta) \mapsto \Theta \).

Can try to extend to a section of assoc. \( \mathbb{R}^2 \)-bundle.

\( (r, \Theta) \mapsto (r, \Theta) \)

There is one zero, at origin. So the cocycle we constructed
for \( S^1 \)-bundles counts intersection pts (with sign) of elts
of \( H^2(X; \mathbb{Z}) \) with themselves.

Using this, and axioms for \( c_1 \) can again show \( e = c_1 \).
**Milnor-Wood Inequality**

Thm. If $E \to \Sigma_g$ is oriented $S^1$-bundle with $g \geq 1$ and has a foliation transverse to the fibers, then $|e(E)| \leq |\chi(\Sigma_g)|$.

Will show: $UT(\Sigma_g)$ realizes this bound.

There is a correspondence:

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{oriented } S^1 \text{-bundles} \\
\text{over } M \text{ with transverse foliation}
\end{array} \right\} & \leftrightarrow \left\{ \tilde{\Pi}_1(M) \to \text{Homeo}^+(S^1) \right\} \\
& \to \text{is monodromy (the foliation identifies pts \& of fibers).} \\
& \leftarrow \text{is: } \tilde{M} \times S^1 / \tilde{\Pi}_1(M) \text{ by diag action gives the bundle, foliation by } \tilde{M} \times \text{pt descends.}
\end{align*}
\]

Unit tangent bundle of $\Sigma_g$. We already know $e(UT(\Sigma_g)) = \chi(\Sigma_g)$.

Need to find foliation.

Setup: $\tilde{\Sigma}_g \cong IH^2 \cong IH^2 \times S^1$ (triv. given by proj. to $\partial IH^2 = S^1$)

\[\tilde{\Pi}_1(\Sigma_g) \to \text{Isom}^+(IH^2) \text{ via deck trans.}\]

induces action on $UT(IH^2)$.

Quotient is precisely $UT(\Sigma_g)$, as desired.

Above theorem due to Wood. Milnor showed if the bundle admits a flat connection (curvature=0) then $|e(E)| \leq |\chi(\Sigma_g)|/2$.

(This is a strictly stronger assumption.)

Later we'll use this to prove $\text{Diff}^+(\Sigma_g) \to \text{MCG}(\Sigma_g)$ has no section.
Pontryagin Classes

Complexification. \( E \to B \overset{\sim}{\to} E^c \to B \)
\( E^c = E \otimes \mathbb{C} \) or \( E \oplus E \) with \( i(x,y) = (-y,x) \).

Pontryagin classes. \( p_i(E) = (-1)^i c_{2i}(E^c) \in H^{4i}(B; \mathbb{Z}) \)

Why only even \( c_i \)? The \( c_{2i+1}(E^c) \) are determined by the \( w_i \):
\( c_{2i+1}(E^c) = \beta(w_{2i}(E)w_{2i+1}(E)) \)
\( \beta \) is Bockstein: \( H^*(G; \mathbb{Z}_2) \)

Relations to other classes. (1) \( p_i(E) \to w_{2i}(E)^2 \) via \( H^{4i}(B; \mathbb{Z}) \to H^{4i}(B; \mathbb{Z}_2) \)
(2) \( p_n(E) = e(E)^2 \quad E = \text{orient. } \mathbb{R}^{2n} \text{- bundle. } \)

Proof. Whitney sum, \( c_2i \to w_{2i}, \quad c_{2n} = e \) later: \( p_i(M^4) \to \tau(M^4) \)

we can now describe all \( \mathbb{Z} \) char classes for real (oriented) bundles.

Thm. (1) \( H^*(G; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \left[ p_1, \ldots, p_{n_{2k}} \right] \)
(2) \( H^*(G; \mathbb{Z})/\text{torsion} \cong \begin{cases} \mathbb{Z} \left[ \tilde{p}_1, \ldots, \tilde{p}_{n_{2k}} \right] & n = 2k+1 \\ \mathbb{Z} \left[ \tilde{p}_1, \ldots, \tilde{p}_{n_{2k}} \right] & n = 2k \end{cases} \)

where \( p_i = p_i(\mathbb{R}^n), \quad \tilde{p}_i = p_i(\mathbb{C}^n), \quad e = e(\mathbb{C}^n) \).

All torsion is 2-torsion, so lies in \( H^*(G; \mathbb{Z}_2) \). It is the image of the Bockstein homomorphism \( \beta: H^*(G; \mathbb{Z}_2) \)

Quick idea: Start with \( 0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0 \)
Apply \( \text{Hom}(G; \mathbb{Z}_4, -) \to \text{LES in } H^* \)
Get \( \beta: H^*(G; \mathbb{Z}_2) \to H^{n+k}(G; \mathbb{Z}_2) \)
(notice \( \deg c_{2i+1} = \deg w_{2i}w_{2i+1} + 1 \).
The computation of $H^*(G_n; \mathbb{Z})$ needs one final tool:

$$\ldots \rightarrow H^{i-n}(B) \xrightarrow{\text{ue}} H^i(B) \xrightarrow{p^*} H^i(SCE)) \rightarrow H^{i-n+1}(B) \rightarrow \ldots$$

This sequence is the LES for $(D(E), S(E))$ in disguise:

$$\ldots \rightarrow H^i(D(E), S(E)) \xrightarrow{j^*} H^i(D(E)) \rightarrow H^i(SCE)) \rightarrow H^{i+1}(D(E), S(E)) \rightarrow \ldots$$

$$\cong \uparrow \Phi = \text{Thom} \quad \cong \uparrow p^* \quad \cong \uparrow \quad \cong \uparrow \Phi = \text{Thom}$$

$$\ldots \rightarrow H^{i-n}(B) \xrightarrow{\text{ue}} H^i(B) \xrightarrow{p^*} H^i(SCE)) \rightarrow H^{i-n+1}(B) \rightarrow \ldots$$

Commutativity of first square.

$$j^* \Phi(b) = j^*(p^*(b) \cup c)$$

$$= p^*(b) \cup j^*(c)$$

$$= p^*(b) \cup p^*(e)$$

$$= p^*(b \cup e).$$

The map $H^i(S(E)) \rightarrow H^{i-n+1}(B)$ is called the Gysin map. It is defined s.t. the third square commutes.

For $B$ a manifold, it can also be defined by:

$$H^i(S(E)) \xrightarrow{p^*} H_{K+n-1-i}(S(E)) \xrightarrow{p^*} H_{K+n-1-i}(B) \xrightarrow{p^*} H^{i+n+1}(B).$$

Or: given an $i$-cochain $\phi$ on $SCE$ we evaluate on an $(i-n+1)$-chain $\sigma$ in $B$ by taking the pullback $S^{n-1}$ bundle over $\sigma$ and applying $\phi$ to this.
Computing with Gysin

The computation of $H^\ast(G_n; \mathbb{Z})$ is modeled on the following argument for $H^\ast(G_n; \mathbb{Z}_2)$.

$E_n \xrightarrow{\pi} G_n$ universal bundle.

$S(E_n) = \{(v, l)\} \quad l = \text{n-plane in } \mathbb{R}^\infty, \quad v \in l \text{ unit.}$

Define $p: S(E_n) \to G_{n-1}$

$(v, l) \mapsto v^\perp \leq l$

This is a fiber bundle, with fiber $S^\infty = \text{unit vectors in } \mathbb{R}^\infty \text{ L to given (n-1)-plane.}$

$S^\infty \text{ contractible } \Rightarrow p^\ast \text{ is } \simeq \text{ on } H^\ast.$

Gysin: $\ldots \to H^i(G_n) \xrightarrow{u^\ast} H^{i+n}(G_n) \xrightarrow{\eta} H^{i+n}(G_{n-1}) \to H^{i+n}(G_n) \to \ldots$

Key step. $\eta(W_j(E_n)) = W_j(E_{n-1}).$

By defn $\eta$ is the composition $H^\ast(G_n) \xrightarrow{\eta^\ast} H^\ast(S(E_n)) \xleftarrow{p^\ast} H^\ast(G_{n-1})$

induced by $G_{n-1} \xleftarrow{p} S(E_n) \xrightarrow{\pi} G_n$

Take pullback $\pi^\ast(E_n) = \{(v, w, l) : l \in G_n, v, w \in l, \ v \text{ unit}\}$

$\cong L \oplus p^\ast(E_{n-1})$

where $L$ is subbundle with $w \in \text{span}(v)$.

$p^\ast(E_{n-1})$ is subbundle with $w \perp v$.

But $L$ is trivial: it has section $(v, v, l)$

So: $\pi^\ast w_j(E_n) = w_j \pi^\ast(E_n) = w_j(L \oplus p^\ast(E_{n-1}))$

$= w_j p^\ast(E_{n-1}) = p^\ast w_j(E_{n-1})$ as desired.

Thus $\eta$ surjective. Now induct on $n$!
Characteristic Classes for Surface Bundles: An Overview

Surface bundles. These are smooth fiber bundles
\[ \Sigma_g \to E \to B \]
\[ \Sigma_g \to E \to B \]
i.e. $B$ covered by $U$ s.t. $p^{-1}(U) \cong U \times \Sigma_g$ (restriction to fibers smooth)

Examples. $B \times \Sigma_g$
- $M_q = \text{mapping torus of } q: \Sigma_g \to \Sigma_g$. \( B = S^4 \)
- $M_q \times S^1 \to T^2$

Isomorphism. As before, a homeo $E \overset{p}{\to} B$ to $E' \overset{p'}{\to} B$ taking $p^{-1}(b)$ to $(p')^{-1}(b)$ by diffeo.

Pullback. As before, given $f: A \to B$, we set
\[ f^*(E) = \{(a, x) : f(a) = p(x)\} \]

Characteristic classes. Fix $g, R$. A char class is a fn
\[ \chi: \{\Sigma_g\text{-bundles}\}/\cong \to H^*(\text{Base}; R) \]
that is natural:
\[ \chi(f^*(E)) = f^* \chi(E). \]

Why? Surface bundles are basic fiber bundles/Manifolds.
Want invariants.
There are other applications to mapping class groups.
We study surface bundles in analogy with vector bundles.

- A Grassmannian for surface bundles

\[ C(\Sigma_g, \mathbb{R}^n) = \text{space of smooth (oriented) submanifolds of } \mathbb{R}^n \text{ diffeo to } \Sigma_g. \]

\[ E(\Sigma_g, \mathbb{R}^n) = \{(x, S) \in \mathbb{R}^n \times C(\Sigma_g, \mathbb{R}^n) : x \in S\} \]

\[ E(\Sigma_g, \mathbb{R}^n) \to C(\Sigma_g, \mathbb{R}^n) \text{ is a } \Sigma_g\text{-bundle.} \]

We will show:

\[ \left\{ \Sigma_g\text{-bundles over } B \right\} / \cong \leftrightarrow \left[ B, C(\Sigma_g, \mathbb{R}^n) \right] \]

and so (fixing \( g, R \)):

\[ \{ \text{char. classes for } \Sigma_g\text{-bundles} \} \leftrightarrow H^* C(\Sigma_g, \mathbb{R}^n). \]

- The mapping class group

In vector bundle case, can reduce structure group to \( O(n) \)

i.e. transition maps can be taken to be isometries on fibers.

Have an analogous reduction here.

\[ \text{MCG}(\Sigma_g) = \text{Diff}^+(\Sigma_g) \]
\[ = \text{Diff}^+(\Sigma_g) / \text{isotopy} \]
We'll show: \( \text{Diff}(\Sigma_g) \) has contractible components, i.e.

\[
\text{Diff}^+(\Sigma_g) \cong \text{MCG}(\Sigma_g)
\]

From this we can deduce:

\[
\left\{ \Sigma_g\text{-bundles} \right\} \leftrightarrow \left[ B, K(\text{MCG}(\Sigma_g), 1) \right] \\
\leftrightarrow \text{Hom}(\pi_1(B), \text{MCG}(\Sigma_g)) / \text{MCG}(\Sigma_g)
\]

and so:

\[
\left\{ \text{char. classes for } \Sigma_g\text{-bundles} \right\} \leftrightarrow H^* \text{MCG}(\Sigma_g).
\]

- **Mori-Tsutsumi-Miller classes.**

  Given \( \Sigma_g \to E \to M = \text{smooth manifold} \)

  Let \( V \) = vertical 2-plane bundle on \( E \)

  Define \( e_i(E) = \text{Gysin}(e_i^*) \in H^{2i}(M) \).

  We'll see: \( e_1 \) is proportional to: Signature, WP form, 1st Pontryagin class.

  \[
  \lim_{g \to \infty} H^*(\text{MCG}(\Sigma_g^+); \mathbb{Q}) \cong \mathbb{Q}[e_1, e_2, \ldots]
  \]

  i.e. the \( e_i \) exactly describe the stable rational char. classes.

- **Unstable classes**

  We know \( \chi(\text{MCG}(\Sigma_g)) = \frac{3}{2}(1-2g)_{2-2g} \). So there are lots of other char. classes. Almost nothing is known.
Cohomology of Mapping Class Groups \( \text{coeff} = \mathbb{Q} \)

**Thm.** \( \text{vcd}(\text{MCG}(\Sigma_g)) = 4g-5 \) \( \Rightarrow \) \( H^i(\text{MCG}(\Sigma_g)) = 0 \) \( i > 4g-5 \)

(although \( H^{4g-5}(\text{MCG}(\Sigma_g)) = 0 \)).

**Low dim's:**
- \( H^1(\text{MCG}(\Sigma_g)) = 0, \ g \geq 0 \).
- \( H^2(\text{MCG}(\Sigma_g)) = \mathbb{Q}, \ g \geq 4 \).
- \( H^3(\text{MCG}(\Sigma_g)) = 0, \ g \geq 6 \).
- \( H^4(\text{MCG}(\Sigma_g)) = \mathbb{Q}^2, \ g \geq 10 \).

**Low genus:**
- \( H^*(\text{MCG}(T^2)) = 0 \).
- \( H^*(\text{MCG}(\Sigma_2)) = \mathbb{Q}[\mathbb{C}^*_2] \).
- \( H^*(\text{MCG}(\Sigma_3)) = \mathbb{Q}[\mathbb{C}^*_2, \mathbb{C}_6] \).
- \( H^*(\text{MCG}(\Sigma_4)) = \mathbb{Q}[\mathbb{C}^*_2, \mathbb{C}_6, \mathbb{C}_9] \).

\[ \text{Cs, C}_6 \text{ unstable.} \]

\[ \text{Also, } H^i(\text{MCG}(\Sigma_g)) \cong H^i(\mathbb{M}(\Sigma_g)) \]

\[ \text{in this case.} \]

**Stability.** \( H^i(\text{MCG}(\Sigma_g)) \) indep of \( g \), \( g > 3i/2 + 1 \).

**Mumford Conjecture.** \( H^i(\text{MCG}(\Sigma_{2n})) = \mathbb{Q}[e_1, e_2, ...] \) \( e_i \in H^{2i} \) \( i^{th} \) MMM class.

**Euler char.** \( \chi(\text{MCG}(\Sigma_g)) = \frac{5(1-2g)}{2} \cdot 2^g \sim (-1)^g \beta(2g-1)/2^{2g-1} \gamma^{2g} \)

\( \Rightarrow \) \( > 2^g \) unstable classes.

\( \text{use: } p(n) \sim \frac{1}{n} e^{\sqrt{2}n^{3/2}} \)

**Applications.**
1. \( \text{Diff}^+(\Sigma_g) \xrightarrow{\eta} \text{MCG}(\Sigma_g) \) has no section
   \( \text{pf: } \eta^*(e_3) = 0 \).
2. Odd \( e_i \) are geometric, cobordism invar, vanish on handlebody group.
A Classifying Space for Surface Bundles

Goal: \( \{ \Sigma_g \text{-bundles over } \mathcal{B} \} / \sim \leftrightarrow [\mathcal{B}, K(\text{MCG}(\Sigma_g), 1)] \)
\( \sim \leftrightarrow \text{Hom}(\pi_1(\mathcal{B}), \text{MCG}(\Sigma_g))/\text{MCG}(\Sigma_g) \)
\( \Rightarrow \text{Ring of char. classes for } \Sigma_g \text{-bundles } \cong H^*(\text{MCG}(\Sigma_g)) \)

We first construct a direct analogue of \( G_n \). Then use contractibility of \( \text{Diff}^o(\Sigma_g) \) to show this is a \( K(\text{MCG}(\Sigma_g), 1) \) -- this part special to \( \Sigma_g \) bundles.

The Grassmannian. \( G_{\Sigma_g} = \text{set of smooth submanifolds of } \mathbb{R}^\infty \text{ diffeo to } \Sigma_g \).
\( G_{\Sigma_g}(\mathbb{R}^n) \) topologized as quotient \( \text{Emb}(\Sigma_g, \mathbb{R}^n)/\text{Diff}(\Sigma_g) \)
\( \text{and } G_{\Sigma_g} = \lim_{\leftarrow} G_{\Sigma_g}(\mathbb{R}^n) \)
\( \text{L } C^\infty \text{ topology} \)

Canonical bundle. \( E_{\Sigma_g} = \{(x, S) \in \mathbb{R}^\infty \times G_{\Sigma_g} : x \in S\} \)

Need to check \( E_{\Sigma_g} \rightarrow G_{\Sigma_g} \) is a \( \Sigma_g \)-bundle.

i.e. if \( S \in G_{\Sigma_g} \) and \( S' \in G_{\Sigma_g} \) is sufficiently close, need a canonical diffeo \( S' \rightarrow S \).

First for \( G_{\Sigma_g}(\mathbb{R}^n) \).

Main idea: if \( S' \) close to \( S \) then \( S' \) is a section of normal bundle \( N \) of \( S \).

\( \text{then } S' \rightarrow S \) is projection in \( N \).

This is because \( S \) is transverse to fibers, which is an open condition, so nearby \( S' \) is transverse to any given fiber, hence to all nearby fibers, hence to all fibers by compactness.

For \( S' \) close enough to \( S \) there is an isotopy of \( S \) to \( S' \) preserving transversality, hence \( S' \cap \text{fiber } = \text{1 pt} \)
\( \Rightarrow S' \text{ a section} \).

The result follows by defn of topology on \( G_{\Sigma_g} \).
Universality. To show \( \{ \Sigma_g \text{- bundles over } B \} / \sim \leftrightarrow \left[ B, G \Sigma_g \right] \quad B = \text{paracompact} \)

Essentially same as v.b. case. Basic idea: Realizing \( E \rightarrow B \) as \( f^* (E \Sigma_g) \) equiv. to finding \( E \rightarrow \mathbb{R}^n \) smooth emb.

on fibers. Such \( g \) induces \( f, \tilde{f} \) s.t.

\[
\begin{array}{c}
E \xrightarrow{f} E_{\Sigma_g} \\
\downarrow \quad \downarrow \\
B \xrightarrow{f} G_{\Sigma_g}
\end{array}
\]

Fix some \( E \rightarrow B \leftarrow \text{compact} \). Want to find \( g \), hence \( f \).

Choose \( U_i \subseteq B \) s.t. \( p^{-1}(U_i) \cong U_i \times \Sigma_g \), part of \( \{ \varphi_i \} \)

\( \varphi_i : p^{-1}(U_i) \rightarrow U_i \times \Sigma_g \rightarrow \Sigma_g \rightarrow \mathbb{R}^n \) \text{ by emb.}\n
\( g : E \rightarrow \mathbb{R}^n \times \cdots \times \mathbb{R}^n \subset \mathbb{R}^\infty \)

\( p \mapsto (\varphi_1 (p), \ldots, \varphi_N (p)) \)

Any two \( g \)'s are homotopic: \( g_0 \xrightarrow{\text{even coords}} g_1 \xrightarrow{\text{odd coords}} \text{ homotopic} \)

\( \sim \) resulting \( f \) unique up to homotopy.

Relation to MCG. Step 1: There is a bundle \( \text{Diff}^\infty (\Sigma_g) \rightarrow \mathcal{P}_g \rightarrow G_{\Sigma_g} \)

(use tubular nbds / sections as above)
\text{Emb}(\mathbb{Z}_g, \mathbb{R}^\infty)

\text{Step 2: } \mathbb{R} = *

Enough to find canonical, continuously varying paths to some basept. \(S\)
Choose \(S\) in even coords.
For any \(S\), apply \(\mathbb{R}^\infty \to \mathbb{R}^{\text{odd coords}}\)
then straight line homotopy to \(S\).

\text{Step 3: Apply L.E.S for fiber bundle (or, fibration)}

\[ \cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \]

(comes from L.E.S. in \(\pi_{*}\) for \((E, F)\) and \(\pi_{*}(E, F) \cong \pi_{*}(B)\)).

\text{Thm (Earle-Eells). } \text{Diff}(\mathbb{Z}_g) \text{ has contractible components.}

\[ \sim \pi_i(G_{\mathbb{Z}_g}) \cong \pi_{i-1}(\text{Diff}(\mathbb{Z}_g)) \quad \forall i. \]

\[ \pi_i(G_{\mathbb{Z}_g}) \cong \pi_{i-1}(\text{Diff}(\mathbb{Z}_g)) = \text{MCG}^+(\mathbb{Z}_g) \]

\[ \pi_i(G_{\mathbb{Z}_g}) = 0 \quad i > 1. \]
Diffeomorphism Groups of Surfaces

$S$ = compact, connected surface
Write $\text{Diff}(S)$ for $\text{Diff}(S, \partial S)$. $C^\infty$ topology.

Thm. If $S \neq S^2, \mathbb{RP}^2, T^2, KB$ then the components of $\text{Diff}(S)$ are contractible.

Note: $\text{Diff}(S^2) \cong \text{Diff}(\mathbb{RP}^2) = SO(3)$
$\text{Diff}(T^2) = T^2$, $\text{Diff}(KB) \cong S^1$.

Proof has 3 steps.  
1. Reduction to case $\partial S \neq \emptyset$
   will show $\pi_i(\text{Diff}(S)) \cong \pi_i(\text{Diff}(S - D^2))$.
2. Inductive step
   will show $\pi_i(\text{Diff}(S)) \cong \pi_i(\text{Diff}(S_1))$.
3. Base case
   $\pi_i(\text{Diff}(D^2)) = 0$  i $\geq 1$.

Step 1. Reduction to case $\partial S \neq \emptyset$.

Fix $x_0 \in \partial S \subseteq S$. Let $S_0 = S - \text{int} D$.
To show $\pi_i(\text{Diff}(S)) = \pi_i(\text{Diff}(S, x_0)) = \pi_i(\text{Diff}(S - D)) = \pi_i(\text{Diff}(S_0))$

Last equality easy. Remains to do other two.
First equality. There is a fiber bundle, $\text{Diff}(S, x_0) \rightarrow \text{Diff}(S) \rightarrow S.
\uparrow \text{diffeos fixing } x_0.$

$\Rightarrow \text{LES:}$

$\pi_{i+1}(S) \rightarrow \pi_i(\text{Diff}(S, x_0)) \rightarrow \pi_i(\text{Diff}(S)) \rightarrow \pi_i(S)$

But $\pi_i(S) = 0$ for $i > 1$ (as $S$ is a finite set).

$\Rightarrow \pi_i(\text{Diff}(S, x_0)) \cong \pi_i(\text{Diff}(S))$ for $i > 1$.

$i=1$ case:

$0 \rightarrow \pi_1(\text{Diff}(S, x_0)) \rightarrow \pi_1(\text{Diff}(S)) \rightarrow \pi_1(S, x_0)$

$\Rightarrow \pi_0(\text{Diff}(S, x_0)) = \text{MCG}(S, x_0)$

Suffices to show $\ker d = 0$.

But the composition

$\pi_1(S, x_0) \rightarrow \text{MCG}(S, x_0) \rightarrow \text{Aut} \pi_1(S, x_0)$

is $\alpha \mapsto$ inner automorphism conj. by $\alpha$.

To show this is inj., suffices to show $\pi_1(S) = 1$.

For latter: $\pi_1(S) \cong \pi_1 H^2$

$\pi_1(S) \leftrightarrow$ deck trans. in $\text{Isom}^+ H^2$

& independent hyperbolic isometries do not commute.

$\triangleright \text{fixed}$

Second equality. Another fiber bundle: $\text{Diff}(S, D) \rightarrow \text{Diff}(S, x_0) \rightarrow \text{Emb}(D, x_0), (S, x_0)$

Claim: $\text{Emb}(D, x_0), (S, x_0) \cong \text{GL}_2(\mathbb{R}) \cong O(2)$

$\Rightarrow \pi_1(\text{Diff}(S, x_0)) \cong \pi_1(\text{Diff}(S, D))$ for $i > 1$. As above, $\text{LES} \Rightarrow \pi_i(\text{Diff}(S, x_0)) \cong \pi_i(\text{Diff}(S, D))$ for $i > 1$. 

\[ i = 1 \text{ case:} \quad 0 \rightarrow \pi_1 \Diff(S, D) \rightarrow \pi_1 \Diff(S, x_0) \]
\[ \rightarrow \pi_1 \text{Emb}((D, x_0), (S, x_0)) \xrightarrow{\partial} \pi_0 \Diff(S, D) = \text{MCG}(S_0). \]
\[ \mathbb{Z} \]

Again, need \( \ker \partial = 0 \).

But \( \mathbb{Z} \rightarrow \text{MCG}(S_0) \rightarrow \text{Aut} \pi_1(S_0, \rho) \)

is \( 1 \rightarrow \text{conj. by } \partial \)-element.

Since \( \pi_1(S_0) \) is free, we are done.

Another point of view. We could have combined the two steps.

There is a fiber bundle
\[ \Diff(S, (p, \nu)) \rightarrow \Diff(S) \rightarrow \text{UT}(S) \]

with fiber \( \pi \Diff(S_0) \).

Apply same argument.
Step 3. Base step: \( \Diff_0(D^2) \) contractible

\[ D_+^2 = \text{top half of } D^2 \]
\[ \Emb(D_+^2, D^2) = \text{space of embeddings } D_+^2 \to D^2 \text{ fixing } D_+^2 \cap \partial D^2 \]
and taking rest of \( D_+^2 \) to \( \text{int } D^2 \).
\[ \kappa = D^1 = \text{equator of } D^2 \]
\[ A(D^2, \kappa) = \text{embeddings of proper arcs in } D^2 \text{ with same endpts as } \kappa. \]

\[ \rightsquigarrow \text{Fibration} \quad \Diff(D_+^2) \to \Emb(D_+^2, D^2) \]

\[ \Downarrow \]

\[ A(D^2, \kappa) \]

Claim 1. \( \Emb(D_+^2, D^2) \sim \sim \ast. \)

Claim 2. \( A(D^2, \kappa) \sim \sim \ast. \)

Uses: the space of tubular nbd's of a submanifold is contractible.

More generally, \( A(S, \kappa) \sim \sim \ast. \) Proven below.

\[ \text{LES} \implies \Diff(D_+^2) \sim \sim \ast. \quad \text{But } D_+^2 \subseteq D^2. \]

Step 2. Induction step.

Induction on \(-\kappa(S)\).

\( \kappa = \text{proper arc in } S. \)

\[ A(S, \kappa) = \text{emb's of proper arcs in } S, \text{ iso to } \kappa, \text{ same endpts} \]

\[ \rightsquigarrow \text{fiber bundle} \quad \Diff_0(S, \kappa) \to \Diff_0(S) \to A(S, \kappa) \]

↑diffeos fixing \( \kappa \) ptwise, \( \sim \Diff_0(S \text{ cut along } \kappa) \).

\[ \text{LES + induction + Claim 2 } \implies \Diff_0(S) \sim \sim \ast. \]
SMALE's Proof. (Original version of Step 3)

The space of \( C^\infty \) diffeos of \( I^2 \) that are id in nbd of \( \partial I^2 \) is contractible.

Some ideas. Given \( f : I^2 \to I^2 \) vector field \( V : \)
\[
V(x,y) = df^{-1}(x,y)(1,0).
\]

There is a homotopy \( V_t \) s.t. \( V_0 = V \), \( V_1 \) = const. vector field \( (1,0) \), \( V_t \) = nonvan. vector field since \( V_0, V_1 : I^2 \to \mathbb{R}^2 - \{0\} \).

id in nbd of \( \partial I^2 \).

Then define, \( f_t : I^2 \to \mathbb{R} \times [0,1] \)
\[
f_t(x,y) = \text{flow along } V_t, \text{ start at } (0,y), \text{ for time } x.
\]

Clearly \( f_1 = \text{id}, f_0 = f \). (n.b. no spiralling, for then there would be a singularity).

Problem: \( \text{Im } f_t \) maybe not = \# \( I^2 \).

Solution: Precompose each \( f_t \) with a reparameterization in the \( x \)-dir. Result is a \( \mathbb{R}^2 \) homotopy of \( f \) to id through diffeos.

By fixing once and for all a retraction of \( \mathbb{R}^2 - \{0\} \) to a point, get a consistent way of deforming an arbitrary diffeo to id, at all times = id in nbd of \( \partial I^2 \).

(See Lurie's notes for an Earle-Eells-style approach.)
CERF'S STRAIGHTENING TRICK.  (Toy case for Claim 2).

We'll need to know that some basic spaces of embeddings are contractible. We start with a warmup.

**Prop.** The space of smooth embeddings of arcs in $\mathbb{R} \times [0, \infty)$ based at 0 is contractible.

**Pf.** The space of linear arcs is clearly contractible — it is homeo to $\mathbb{R} \times [0, \infty)$.

Here is a canonical isotopy from an arbitrary arc $f$ to a linear one:

$$F_t(x) = \begin{cases} f((1-t)x) & t < 1 \\ \frac{1-t}{1-t} & t = 1 \\ f'(0)x & t = 1 \end{cases}$$

Can soup this up:

**Prop.** The space of smooth embeddings of arcs in $S$ based at $p \in S$ is contractible.

**Pf.** By previous prep., need a canonical isotopy of an arbitrary arc into a fixed tubular nbhd of $p$.

For any compact set of arcs, can use

$$F_t(x) = f(\kappa x) \quad \kappa = \max \{ \epsilon, (1-tx) \}$$

i.e. $F_t(x)$ traces out shorter & shorter subarcs.

This implies weak contractibility.
Claim 2: Contractibility of arc spaces

\( \alpha = \text{proper arc in } S \)
\( A(S, \alpha) = \text{space of proper arcs } \sim \alpha, \text{ same endpoints as } \alpha. \)

Case 1. \( \alpha \) connects distinct components of \( \partial S \).

\( T = \text{surface obtained from } S \text{ by capping with disk at one end of } \alpha \)

\( \sim \text{ fiber bundle } \quad \text{Emb}(I, S) \to \text{Emb}(IUD^2, S) \)
\( \uparrow \quad \downarrow \quad \text{both endpoints fixed} \quad \text{Emb}(D^2, T-\partial T) \)

Claim. \( \text{Emb}(IUD^2, S) \cong * \)

pf of claim. Another fiber bundle \( \text{Emb}(D^2, S) \to \text{Emb}(IUD^2, S) \)

Base, fiber contractible by variations on Cerf's straightening.

Claim. \( i_* \text{Emb}(D^2, T-\partial T) \cong 0 \)

pf. Yet another fiber bundle:
\( \text{Emb}(D^2, T-\partial T) \)
\( \downarrow \text{eval} @ 0 \)
\( T-\partial T \)

By two claims, plus LES for main fiber bundle, \( \text{Emb}(I, S) \) has contractible components, one of which is \( A(S, \alpha) \).
Case 2. \( \alpha \) joins a component of \( \partial S \) to itself.

Idea: add a handle \( T = SU \) s.t. 
\( \alpha \) joins distinct comps of \( \partial T \)
Suffices to show 
\[ \pi_1 A(T-\beta) \rightarrow \pi_1 A(T) \] injective.

Key: there is a cov. space of \( T \) hom. eq. to \( S \),
because \( \pi_1(T) = \pi_1(S) \ast \mathbb{Z} \)
So \( \tilde{T} \) = cover corr to \( \pi_1(S) \)

\( \tilde{T} \) looks like

\( T \) cut along \( \beta \)

Identify \( A(T-\beta, \alpha) \) with space of arcs in this region of \( \tilde{T} \).
\( A(T, \alpha) \) with space of arcs in \( \tilde{T} \):

\[ \text{lifts of arcs in } T \rightarrow \tilde{A}(T, \alpha) \subseteq A(T, \alpha) \leftarrow \text{arcs in } T \]

Suffices to show composition \( A(T-\beta, \alpha) \rightarrow A(T, \alpha) \) is inj on \( \tilde{T} \).

Need a retraction \( r : A(\tilde{T}, \alpha) \rightarrow A(T-\beta, \alpha) \)

s.t. \( r \circ i = \text{id} \).

The \( r \) is induced by shrinking the two contractible pieces.
CHARACTERISTIC CLASSES IN DEGREE ONE

We know now: $H^*(\text{MCG}(S_g)) \cong \text{Ring of char classes for } S_g\text{-bundles}

Thm. $H^i(\text{MCG}(S_g); \mathbb{Z}) = 0$ $g > 1$.

pf. We'll do $g > 3$. Ingredients:

1. $\text{MCG}(S_g)$ is gen. by Dehn twists about nonseparating curves

2. Any two such Dehn twists are conjugate in $\text{MCG}(S_g)$

3. There is a relation among such twists of the form

$$T_x T_y T_z = T_a T_b T_c T_d$$

It follows that $H_1(\text{MCG}(S_g); \mathbb{Z}) \cong \text{MCG}(S_g)^{ab}$ is trivial.

hence $H^i(\text{MCG}(S_g); \mathbb{Z}) = 0$.

Ingredient 2. Follows from: $f T a f^{-1} = T f(a)$ and classification of surfaces.

Ingredient 3. Follows from: Lantern relation

(prove by checking action on $\mathbb{Z}$ and using $\text{Mod}(\mathbb{Z}^3) = 1$)

and the embedding:
GENERATING MCG (Ingredient 1).

Two ingredients: 1. The complex of curves $C(S_g)$ is connected $g \geq 2$.
   vertices: isotopy classes of simple closed curves
   edges: disjoint representatives

2. The Birman exact sequence $\gamma_1(S) < 0$.
   $1 \to \pi_1(S, p) \to \text{MCG}(S, p) \to \text{MCG}(S) \to 1$.

Outline of proof. 1. $\Rightarrow$ complex of nonsep. curves $N(S_g)$ is connected.
   $\Rightarrow$ given any two isotopy classes of nonsep s.c.c. in $S_g$
   $\exists \ C T c_i; c_i$ nonsep taking one to other.*
   $\Rightarrow$ MCG($S_g$) gen. by nonsep twists if
   MCG($S_g - c$) is.
   But MCG($S_g - c$) $\cong$ MCG($S_{g-1}$)
   (applied twice).
2. $\Rightarrow$ MCG($S_{g-1}$) is gen by nonsep twists
   if MCG($S_{g-1}$) is.

Done by induction. Base case is MCG($S_1$) $\cong SL_2 \mathbb{Z}$
   gen by $(0,1), (-1,0)$.

* Use the relation $T_b T_a(b) = a$ for $i(a,b) = 1$. 

\[ \]
Connectivity of $C(S_g)$

Take two vertices of $C(S_g)$, represent them by s.c.c. in $S_g$.
Choose smooth fns $f_0, f_1$, s.t. $a$ is a level set of $f_0$, $b$ of $f_1$.
Connect $f_0$ to $f_1$ by a path $f_t \in C^\infty(S_g, \mathbb{R})$.

Cert Lemma. Any path $f_t \in C^\infty(S_g, \mathbb{R})$ can be approx. by $g_t \in C^\infty(S_g, \mathbb{R})$
so each $g_t$ is in one of the following classes:

1. Morse functions with at most 2 coincident critical values $\leftrightarrow$ crit. values passing each other
2. Functions with distinct crit vals and exact one
   degen. crit pt of the form $x^3 + y^2 + c$ $\leftrightarrow$ crit vals merging/splitting

Claim. Each $g_t$ has a level set rep. a vertex of $C(S_g)$.

Notice curves are isotopic $\Rightarrow \{ t : v \in C(S_g) \}$ is rep by a level set of $g_t$.
which is open in $\mathbb{R}$

Also, level sets of the same $g_t$ are disjoint.
Result follows from compactness of $[0,1]$.

Remains to prove claim. Take nbd of crit set: $\square$

If two circles bound disks, modify the function to get rid of this crit pt.
Look at another crit pt.

Or: Given $f : S_g \to \mathbb{R}$ $\sim$ graph $\Gamma_f$ by crushing level sets.
$\text{rk}(\Gamma_f) = g$. Except in case 2 above where $\text{rk}(\Gamma_f) = g-1$.
Any nontrivial cocycle (= pt) in $\Gamma_f$ corresponds to a nontrivial level set in $S_g$. (this shows $N(S_g)$ connected!)
\[ S_g \rightarrow E \rightarrow B \]
\[ \sim \quad V = \text{vertical 2-plane bundle on } E. \]

\[ e_1(E) = \text{Gysin} \left( e(V)^2 \right) \in H^4(B). \]

For \( B = S_h \), compute by intersecting 2 generic sections with \( O \)-section, since
1. \( e \) is P. dual to \( \text{section} \cap O \)-section
2. \( U \) is P. dual to \( \cap \)
3. Gysin is P. dual to projection.

We will see: if \( E_1 \) diffeo. \( E_2 \) then \( e_1(E_1) = e_1(E_2) \)

\[ \text{e.g. Atiyah-Kodaira: } S^4 \rightarrow M \quad S^4 \rightarrow M \]
Say \( e_1 \) geometric.

\[ S^4 \rightarrow S^1 \quad S^2 \]

More generally: \( e_i(E) = \text{Gysin} \left( e(V)^{i+1} \right) \in H^{2+i}(B) \)

Compute by intersecting \( i+1 \) sections with \( O \)-section.

Thm. (Church-Farb-Thibault) \( e_{2i+1} \) geometric.

Want to show \( e_i \not\equiv 0 \). Need \( S_g \rightarrow M^{2i+2} \rightarrow B^{2i} \) with \( e_i(M) \not\equiv 0 \quad \forall \ g, i \).

Will use branched covers.
SIGNATURE

\[ M = \text{closed, oriented 4k-manifold} \]

\[ \rightarrow H^{2k}(M; \mathbb{Q}) \otimes H^{2k}(M; \mathbb{Q}) \rightarrow H^{4k}(M; \mathbb{Q}) \cong \mathbb{Q} \]

\[ \alpha \otimes \beta \mapsto \alpha \cup \beta \]

bilinear form, symmetric since \( 2k \) even.

\( \varphi(M) = \text{signature of this form} : \# \text{pos. eigen vals} - \# \text{neg. eigen vals} \)

Rochlin: \( \varphi(M^+) = 0 \iff M^+ = \partial W^5 \)

Hirzebruch: \( p_1(M^+) = 3\varphi(M^+) \) (baby case of H.T formula)

\[ \text{Prop.} \quad S_q \rightarrow E \rightarrow S_h \]

\[ \Rightarrow \langle e_i(E), S_h \rangle = \langle p_1(E), E \rangle \quad (= \# 3\varphi(E)) \]

Cor: \( e_1 \) is geometric.

\[ \text{Pf of Prop.} \; \; \; \; \; TE \cong V \oplus \mathfrak{n}^* S_h \]

\[ \Rightarrow p_1(E) = p_1(V \oplus \mathfrak{n}^* S_h) \]

\[ = p_1(V) + \mathfrak{n}^* p_1(S_h) \]

\[ = e(V)^2 + 0 \quad \text{in general } p_1 = e^2 \]

\[ \Rightarrow \langle e_i(E), S_h \rangle = \langle \text{Gysin}(e(V)^2), S_h \rangle \]

\[ = \langle e(V)^2, E \rangle \]

\[ = \langle p_1(E), E \rangle \]

\[ \text{exercise:} \]

\[ (G) \quad \text{Gysin}(\alpha(eV)) = \alpha(\mathfrak{n}^* S_h) \]

\[ (6) \quad \mathfrak{n}^* S_h = E \]
Branched Covers

A cyclic branched cover is a map $\tilde{M} \to M$ that is a cyclic covering away from a codim 2 subman of $M$ = ramification locus (can allow more complicated ram. locus, but we won't)

A $p \in M \ni$ nbd $U$ s.t. $p^{-1}(U) \to U$ is
1. trivial $m$-fold cover ($m$ copies of $U$), or
2. quotient by order $m$ rotation ($m = \text{degree of cover}$)

E.g.

Can sometimes get cyclic branched covers via group actions: Say $\mathbb{Z}/m \subset N$ by or. pres. differs s.t.
1. fixed set has codim 2, $F$ = mnfld
2. action free outside $F$

Then $\bar{N} = N/\mathbb{Z}/m$ is a manifold (check!) and $N \to \bar{N}$ is cyclic b.cover
Near $F$, proj looks like $F \times \mathbb{C} \to \bar{F} \times \mathbb{C}$
$(p, z) \mapsto (p, z^m)$

Thm. Every closed, or. 3-man is a 3-fold branched cover over $S^3$. 

Simple
Existence of Branched Covers

Prop. \( M = \text{closed or smooth} \) \( \mathbb{R} \)-man.
\( B \subseteq M \) or subman of codim 2.
If \( [B] \in H_{n-2}(M) \) divs. by m. in \( H_{n-2}(M; \mathbb{Z}) \),
then \( \exists \) m-fold cyclic branched cover over M
ramified along B.

Proof for \( M = S^3, B = K \). Let \( S = \text{Seifert surface} \)
\( \mapsto [S] \in \pi_2 H_2(S^3, K) \)
\( \cong H^1(S^3 - K) \)

(via \( H_2(S^3, K) \to H_2(S^3 - K, \mathbb{Z}) \to H_2(S^3 - N(K), \mathbb{Z}) \))
\( \xrightarrow{\text{PD}} H^1(S^3 - N(K)) \to H^1(S^3 - K) \)

The elt of \( H^1 \) is signed intersection with \( S \).
An elt of \( H^1(S^3 - K) \) is a map \( H_1(S^3 - K) \to \mathbb{Z} \).
Reduce mod any m, get a cover over \( \# S^3 - K \).
Glue \( K \) into the cover.

This works in general. There is no \( \# \text{Seifert surface per se} \),
but there is a \( \# \) class in \( H_{n-1}(M, \mathbb{Z}) \) with boundary \( B \).
Then, elts of \( H^1(M; \mathbb{Z}) \) are maps \( H_1(M; \mathbb{Z}) \to \mathbb{Z} \), so
\( \$ \) can proceed as above.

We know the elt of \( H^1 \) is nontrivial by considering a small
loop around \( B \) in \( M \). It intersects \( A \) in one pt.
Existence of Branched Covers II

Vector Bundle Version.

Suppose \([B] = m[A]\) in \(H_n(M; \mathbb{Z})\)


We know:

\[
\text{Group of } C^1\text{-bundles on } M \text{ under } \otimes \cong H^2(M; \mathbb{Z})
\]

Let \(E_B\) be \(C^1\)-bundle corr. to \([B]^*\). This means

\(E_B\) has a section \(s : M \to E_B\) s.t.

\(\text{Im}(s) \cap M = B\).

Similarly, \(E_A \leftrightarrow [A]^*\). By above isomorphism:

\[
E_A^\otimes m \cong E_B
\]

Define

\[
f : E_A \to E_B
\]

\(V \mapsto V \otimes \ldots \otimes V = V^m\)

Set

\[
\tilde{M} = f^{-1}(\text{Im}(s))
\]

Each pt of \(M - B\) has \(m\) preimages: the \(m\)th roots.
Branched Covers and Euler Classes

A cyclic branched cover \( \tilde{E} \xrightarrow{p} E \) is a cyclic branched cover of surface bundles if the restriction of \( p \) to a (surface) fiber is a branched cover of surfaces onto a fiber of \( E \).

Equivalently, \( \tilde{E} \) is a cyclic branched cover over \( E \) s.t. ramification locus intersects each fiber of \( E \) in a \( 0 \)-manifold. (use: the restriction of a (branched) cover to a submanifold of base is a branched cover.)

Prop. Let \( \tilde{E} \xrightarrow{p} E \) be a fiberwise cyclic branched covers over \( M \) with fiber genus \( \frac{2g}{3} \) & \( g \). Then

1. \( p^* \left[ \mathcal{D} \right]^* = 2 \left[ \tilde{\mathcal{D}} \right]^* \quad \mathcal{D} = \text{ram. locus} \)
2. \( e(\tilde{\mathcal{V}}) = p^* e(\mathcal{V}) - \left[ \tilde{\mathcal{D}} \right] \)

Note: (1) is just a fact about branched covers.

Pf of (1). \( p^* \left[ \mathcal{D} \right]^*/\left[ \mathcal{D} \right]^* \) computed \( \chi(\mathcal{D})/\chi(\mathcal{D}) \). Clear when \( \mathcal{D} \) is a \( 0 \)-manifold. In general, replace fundamental class with Thom class of normal bundle.
Pf of (2). Clearly:

\[ H^2(E) \xrightarrow{p^*} H^2(\tilde{E}) \]

\[ \downarrow \quad \cup \quad \downarrow \]

\[ H^2(E \setminus \text{Int}N(D)) \rightarrow H^2(\tilde{E} \setminus \text{Int}N(\tilde{D})) \]

(check on the level of bundles).

\[ \Rightarrow e(V), e(\tilde{V}) \text{ have same image in lower right}. \]

Consider LES of pair:

\[ \cdots \rightarrow H^2(\tilde{E}, \tilde{E} \setminus \text{Int}N(\tilde{D})) \rightarrow H^2(\tilde{E}) \rightarrow H^2(\tilde{E} \setminus \text{Int}N(\tilde{D})) \rightarrow \cdots \]

Since \( p^*e(V), e(\tilde{V}) \) have same image in \( H^2(\tilde{E}, \tilde{E} \setminus \text{Int}N(\tilde{D})) \neq H^2(N(\tilde{D}), \partial N(\tilde{D})) \)

\[ \cong H_{n-2}(\tilde{D}) \cong \mathbb{Z}. \]

Remains to compute this integer. Evaluate \( p^*(e(V)) + k[\tilde{D}]^* \)

and \( e(\tilde{V}) \) on fiber of \( \tilde{E} \):

\[ e(\tilde{V})_{|S_2} = 2 - 2(2g) = 2 - 4g. \]

since fibers map with degree 2.

\[ p^*(e(V))(S_2) = 2(2-2g) = 4 - 4g \]

\[ k[\tilde{D}]^*(S_2) = 2k \quad \leftarrow \tilde{D} \text{ intersects each fiber in 2 pts} \]

\[ \Rightarrow 2 - 4g = 4 - 4g + 2k \]

\[ \Rightarrow k = -1, \text{ as desired}. \]
Thm. \( \tilde{E} \to E \) as above. Then:

\[
e_1(\tilde{E}) = 2e_1(E) - 3i(\tilde{D}, \tilde{D})
\]

Pf. By Prop(2):

\[
e(\tilde{V}) = \rho^*(e(V)) - [\tilde{D}]^*
\]

Squaring:

\[
e(\tilde{V})^2 = \rho^*(e(V)^2) - 2\rho^*(e(V))[\tilde{D}]^* + [\tilde{D}]^*^2
\]

Use Prop(1) \( \Rightarrow \)

\[
e_1(\tilde{E}) = 2e_1(E) - 2(e(\tilde{V})[\tilde{D}]^* + [\tilde{D}]^2) + [\tilde{D}]^2
\]
\[
= 2e_1(E) - i(\tilde{D}, \tilde{D}) - 2e(\tilde{V})[\tilde{D}]^*
\]

Remains to show: \( e(\tilde{V})[\tilde{D}]^* = i(\tilde{D}, \tilde{D}). \)

But since \( \tilde{V} \) is transverse to \( \tilde{D} \) at all points, its restriction to \( \tilde{D} \) is isomorphic to the normal bundle \( N\tilde{D} \)

\[
\Rightarrow e(\tilde{V})[\tilde{D}]^* = e(\tilde{V})(\tilde{D}) = e(N\tilde{D})(\tilde{D}) = i(\tilde{D}, \tilde{D}). \]
Atiyah's Construction

Will form a 2-fold branched cover over $S_{129} \times S_3$.

\[ \text{need a } D \text{ with } [D] \text{ even.} \]

Start with two covers: $S_{129}$

\[ \begin{align*}
  &f \downarrow \\
  &\pi_1(S_3) \rightarrow H_1(S_3; \mathbb{Z}_2) \\
  &\text{cover corr. to} \\
  &\left\langle \pi \right\rangle \\
  &\text{quotient by } \left\langle \pi \right\rangle \\
&h \downarrow \\
S_3 &\rightarrow S_2
\end{align*} \]

$D$ is union of two graphs in $S_{129} \times S_3$:

\[ \begin{align*}
  &\Gamma_f \\
  &\text{"key" will } \Rightarrow [D] \text{ is even} \\
S_3 &\rightarrow S_{129}
\end{align*} \]

Some features:

1. $\Gamma_f \cap \Gamma_{\pi} = \emptyset$ since $\pi$ has no fixed pts
2. Vertical bundle $V$ (= pullback of $TS_3$ via proj to $S_3$) is transverse to $D$
3. Projection $D \rightarrow S_3$ is a covering map (namely $f$).
4. Each $S_3$-fiber intersects $D$ in two pts.

\[ \begin{align*}
  &\Rightarrow V|_D \cong ND \text{ normal bundle} \\
  &\Rightarrow V|_D \cong TD \text{ tangent bundle} \\
  &\Rightarrow i(D,D) = 2i(\Gamma_f, \Gamma_f) \\
  &\Rightarrow \text{when we take the branched cover over } D, \text{ fibers are } S_6.
\end{align*} \]
Claim 0: \([D]\) is even.

Let \([D]^*\) be \(P\)-dual, \(\mathbb{Z}_2\)
\([D]^*_2 \in H^2(S_{129} \times S_3)\)
the mod 2 reduction
Need \([D]^*_2 = 0\).

\[
\begin{align*}
\Delta & \quad \Delta \\
S_{129} \times S_3 & \xrightarrow{(f \times \text{id})} S_3 \times S_3 \xrightarrow{h \times h} S_2 \times S_2 \\
\end{align*}
\]

\([D]^*_2 = (f \times \text{id})^* (h \times h)^* [\Delta]^*\)

But \(H^2(S_2 \times S_2) \cong H^2(S_2 \times \text{pt}) \oplus (H^1(S_2) \otimes H^1(S_2)) \oplus H^2(\text{pt} \times S_2)\)
and \((h \times h)^*\) kills \(H^2\) factors since \(h\) has deg 2
\((f \times \text{id})^*\) kills middle factor since
\(f_*(H_1(S_{129}; \mathbb{Z})) \leq 2 H_1(S_3; \mathbb{Z})\) by defn.

Thus \(E\) 2-fold cyclic branched cover \(E \rightarrow S_{129} \times S_3\)
with ram. locus \(D\).
\(E\) has the structure of a surface bundle over \(S_{129}\)

Fiber is \(S_6\):
Thm. \( e_1(E) = 7e_8 + 0 \).

Pf. By previous Thm.: 
\[
e_1(E) = 2e_1(S_{12g} \times S_3) - 3i(D, D) \\
= -3i(D, D) \\
= -3 \cdot \frac{1}{2} i(D, D) \quad \text{by Prop.(1)} \\
= -3i(\mathcal{F}, \mathcal{F})
\]

Recall from above that the normal bundle \( N\mathcal{F} \) is isomorphic to the tangent bundle \( T\mathcal{F} \) (both are \( \cong \) to \( V\mathcal{F} \)).

So:
\[
i(\mathcal{F}, \mathcal{F}) = e(N\mathcal{F}) = e(T\mathcal{F}) = \chi(\mathcal{F}) = \chi(S_{12g}).
\]
**Construction of $S^g$-Bundles with $e_i \neq 0$**

\[ \begin{align*}
\tilde{E}^* &\to E_3 \to E_2 \to (E_1^*)' \to E_1^* \to E^* \to E \\
\downarrow S_g^n &\quad \downarrow S_g^l \quad \downarrow S_g^l \quad \downarrow S_g^l \quad \downarrow S_g^l \quad \downarrow S_g^l \quad \downarrow S_g \\
S_g &\quad \downarrow S_g^l \quad \downarrow S_g^l \quad \downarrow S_g^l \quad \downarrow S_g^l \quad \downarrow S_g^l \quad \downarrow S_g \\
E &\quad \tilde{E} \quad E_2 \quad E_1 \quad E_1 \quad E \quad M
\end{align*}\]

- **Induction:** Start with $e_i(E) \neq 0$.
- **Dim:** $\dim(M) = 2i$

**Steps:**
1. **Diagonal construction:**
   - Bump up dim by 2,
   - New bundle has diagonal section $\Delta$.
2. **Take cover over base so pullback bundle has $m$-fold fiberwise cover.
3. **Kill action of $\pi_1$ of base on $H^1(fiber; \mathbb{Z})$ and ensure $H^1(E_i) \to H^1(E_2)$ is 0 mod $m$.**
4. **Now can take cover of base so preimage of $\Delta$ is divisible by $m$.**

Morita calls this the $m$-construction on $E \to M$.
Atiyah's construction is $\tilde{E}$-construction on $S_g \to \text{pt}$.

**Prop:**
\[ e_i(\tilde{E}^*) = -dm^2(1 - m^{i+1}) e_i(E) \quad d = \text{degree of } \tilde{E} \to E \]

The proof is analogous to that of: $e_i(\tilde{M}) = 2e_i(M) - i(\tilde{D}, \tilde{D})$ above.

So $e_i(E) \neq 0 \Rightarrow e_i(\tilde{E}^*) \neq 0.$
Higher Dimensional Surface Bundles

Goal. $e_i \neq 0 \ \forall i$.

Iterated surface bundles. $C_0 = \{\ast\}$

$C_{i+1} = \{\text{finite covers of } S_g \text{-bundles over elts of } C_i, g \geq 2\}$

e.g. $C_1 = \{S_g : g \geq 2\}$

Choose $E \in C_i$ surf. bundle with $e_i(E) \neq 0$. $\triangleright$ note: $e_0$ always $\neq 0$, which is why you can use the trivial bundle in Atiyah's construction.

Will use to construct $\tilde{E} \in C_{i+1}$ with $e_{i+1}(\tilde{E}) \neq 0$.

Step 1. $C_i \rightarrow C_{i+1}$

Given $S_g$-bundle, $\pi : E \rightarrow M$

$\sim \quad E^* = \pi^*(E) = \{(u,u') \in E \times E : \pi(u) = \pi(u')\}$

Bundle structure: $\pi' : E^* \rightarrow E$

$(u,u') \mapsto u$

Have a bundle map:

$$
\begin{array}{ccc}
E^* & \xrightarrow{q} & E \\
\pi' \downarrow & & \downarrow \pi \\
E & \xrightarrow{\pi} & M
\end{array}
$$

$E^*$ comes with a section $\Delta = \{(u,u)\}$, which intersects each fiber in one point.

Write $v$ for $\Delta^* \in H^2(E^*; \mathbb{Z})$

$v_m \in H^2(E^*; \mathbb{Z}_m)$ the mod $m$ reduction
Example. \( E = S^g, \ M = \ast. \)
\[ \Rightarrow E^* = S^g \times S^g, \ \Delta = \text{usual diagonal}. \]

Step 2. Given an \( S^g \)-bundle \( E \to M \)

\( \exists \) finite cover \( M_1 \overset{p}{\to} M \)

such that \( p^*(E) \) admits \( m \)-fold (unbranched) cover along fibers.

Note. Step 2 not needed in \( \ast \) case since \( S^g \times S^g \to S^g \) admits \( m \)-fold cover over fibers for any \( m \).

Proof. Pick any \( m \)-fold \( \tilde{S}^g \to S^g \)

Denote \( h : M \to \text{MCG}(S^g) \) the monodromy.

Goal: Construct a cover \( \tilde{M} \to M \) and a monodromy \( \tilde{h} : \tilde{M} \to \text{MCG}(\tilde{S}^g) \) such that \( \tilde{h}(\tilde{x}) \) is a lift of \( h(x) \) \( \forall \tilde{x} \in \pi_1(\tilde{M}). \)

Then check: the combination of the two covering maps (of base and fiber) give a covering map of bundles.*

Need two facts about \( \text{MCG} \):
1. \( \text{Out} \pi_1(S^g) = \text{MCG}^+(S^g) \)
2. \( \text{MCG}(S^g) \) has torsion-free subgroup of finite index, e.g., \( \ker(\text{MCG}(S^g) \to \text{Sp}(2g, Z_3)) \)

* In general, \( \text{pullback}^* \) is given by composition of \( f^* \) (on \( \pi_1 \)) with original monodromy.

Cover along fibers given by lifting monodromy to \( \text{MCG} \) of cover.
Choose $\tilde{\Gamma} \leq \text{Aut } \tilde{\pi}(\tilde{S}_g)$ finite index, preserves $\pi_1(\tilde{S}_g)$

$\rightsquigarrow \Gamma : \tilde{\Gamma} \rightarrow \text{Aut } \tilde{\pi}(\tilde{S}_g) \rightarrow MCG(\tilde{S}_g)$

Note: $r(\tilde{\Gamma} \cap \text{Inn } \tilde{\pi}(\tilde{S}_g))$ consists of torsion since any $x \in \pi_1(\tilde{S}_g)$ has a power in $\tilde{\pi}(\tilde{S}_g)$, which then is an inner auto of $\tilde{\pi}(\tilde{S}_g)$.

$\Rightarrow \exists \tilde{\Gamma}_2 < \tilde{\Gamma}_1$ finite index s.t. $\tilde{\Gamma}_2 \cap \text{Inn } \tilde{\pi}(\tilde{S}_g) = 1$.

(using $\otimes$ above).

$\Rightarrow \Gamma_2 = \pi_1(\tilde{\Gamma}_2)$ finite index in $MCG(S_g)$

$\Rightarrow \Gamma_3 < MCG(S_g)$ finite index (intersect all conjugates of $\Gamma_2$) and $\Gamma_3 \rightarrow MCG(S_g)$ is well defined.

Let $\tilde{M} \rightarrow M$ be the cover given by

$$\pi_1(M) \rightarrow MCG(S_g)/\Gamma_3$$

Then $\tilde{h} : \tilde{M} \rightarrow MCG(\tilde{S}_g)$ given by

$$\pi_1(\tilde{M}) \rightarrow \Gamma_3 \rightarrow MCG(\tilde{S}_g).$$

In other words, we showed: Given $S_g \rightarrow S_g$, $\exists$ finite index $\Gamma < MCG(S_g)$ and a $\Gamma \rightarrow MCG(S_g)$ where each $f \in \Gamma$ maps to a lift of $f$.

Then if the original bundle $E$ has monodromy $p : \pi_1(M) \rightarrow MCG(S_g)$, the monodromy cover of $M$ is the one corresponding to $p^{-1}(\Gamma)$ and the monodromy after taking the fiberwise cover is $p^{-1}(\Gamma) \rightarrow \pi_1(M) \rightarrow \Gamma \rightarrow MCG(S_g)$. 
Step 3. \( E \in C_n, \Delta \in H^2(E) \) \text{ all coeff. } \mathbb{Z}/m\mathbb{Z}

Then \( \exists \text{ finite cover } \tilde{E} \xrightarrow{p} E \) s.t. \( p^*(\Delta) = 0 \).

Induct on \( n \).
Reduce to case. \( E = S^g \)-bundle by taking pullbacks.
Apply Step 2, then take \( m \)-fold fiberwise cover.
Take another pullback to kill action on \( H^1(\text{fiber}) \) and kill \( H^1(\text{base}) \)

\[
\begin{align*}
E_2^* \rightarrow (E^*)' \rightarrow E_1^* \rightarrow E \quad & \quad \Delta \quad \text{Denote} \quad E_2^* \rightarrow E \\
\pi \downarrow S^g' \quad \downarrow S^g' \quad \downarrow S_3 \quad \downarrow S_3 \\
\vee E_2 \rightarrow E_1 \rightarrow E_1 \rightarrow M \quad & \quad \text{by } p_0^* \end{align*}
\]

Claim: \( \exists v \in H^2(E_2) \) s.t. \( p_0^*(\Delta) = \pi^*(v) \)

Pf: Sere spectral seq. (below)

By induction, \( \exists \text{ finite cover } \tilde{E} \rightarrow E_2 \) s.t. \( v \rightarrow 0 \)
in \( H^2(\tilde{E}) \):

\[
\begin{array}{c}
E_3^* \rightarrow E_2^* \\
\downarrow \quad \downarrow \\
\tilde{E} \rightarrow E_2 \\
\end{array}
\]

By commutativity, the result follows.
SERRE SPECTRAL SEQUENCE

Want to prove claim. Write $F \rightarrow E \rightarrow B$ for $Sg \rightarrow E^{2*} \rightarrow E_{2}$

Page 2 of Serre SS:

By construction, all $Z/m$ coeffs are trivial.

$\begin{align*}
H^0(B; H^2(F)) & \rightarrow H^1(B; H^3(F)) \rightarrow H^2(B; H^2(F)) \\
H^0(B; H^1(F)) & \rightarrow H^1(B; H^2(F)) \rightarrow H^2(B; H^1(F)) \\
H^0(B; H^0(F)) & \rightarrow H^1(B; H^0(F)) \rightarrow H^2(B; H^0(F))
\end{align*}$

The Serre SS package gives two things:

1. There is a filtration $F_2 \subseteq F_1 \subseteq F_0 = H^2(E)$ s.t.
   $F_i / F_{i+1} \cong E^{2,2-i}_{\infty}$

2. The map
   $H^2(E) \rightarrow E^{0,2}_{\infty} \rightarrow E^{0,2}_{2} = H^2(F)$
   is the one induced by $F \rightarrow E$.
   (the map $H^2(E) \rightarrow E^{0,2}_{\infty}$ comes from 1, the other map comes from the SS)

What are the $F_i$?

$F_2 / F_3 = F_2 = \frac{1}{4} E^{2,0}_{\infty}$

$F_1 / E^{2,0}_{\infty} = E^{1,1}_{\infty}$

$H^2(E) / F_1 = E^{0,2}_{\infty}$

Still need to determine $F_1$. Have:

$1 \rightarrow F_1 \rightarrow H^2(E) \rightarrow E^{0,2}_{\infty} \rightarrow 1$

The term $E^{0,2}_{\infty}$ is a subgroup of $E^{0,2}_{2}$ (it is the kernel of the differential shown above). So by 2,

$F_1 = K = \ker (H^2(E) \rightarrow H^2(F))$
In other words, we have two short exact seqs:

\[
1 \rightarrow K \rightarrow H^2(E) \rightarrow E_\infty^{0,2} \rightarrow 1
\]

\[
1 \rightarrow E_\infty^{0,0} \rightarrow K \rightarrow E_\infty^{1,1} \rightarrow 1 \quad \leftarrow \text{typo in Morita!}
\]

Recall, we have \( p_0^*(\Delta) \in H^2(E) \), we want to show it lives in \( E_\infty^{0,0} = H^2(B) \).

**Step 1.** Image of \( p_0^*(\Delta) \) in \( E_\infty^{0,2} \) is 0, i.e. \( p_0^*(\Delta) \in K \).

Recall we took an \( m \)-fold fiberwise cover

\[
\begin{array}{ccc}
S_g' & \rightarrow & S_g \\
\downarrow & & \downarrow \\
E_2^* & \rightarrow & E^*
\end{array}
\]

The map \( H^2(S_g) \rightarrow H^2(S_g') \) is **zero**.

The map \( H^2(E_2^*) \rightarrow E_\infty^{0,2} \) is the map \( H^2(E_2^*) \rightarrow H^2(S_g') \)

Use commutativity.

**Step 2.** Image of \( p_0^*(\Delta) \) in \( E_\infty^{1,1} \) is 0, i.e. \( p_0^*(\Delta) \in E_\infty^{2,0} = H^2(B) \)

Recall we arranged \( \theta \) s.t. \( H'(E) \rightarrow H'(E_2) \) is **zero**.
**Algebraic Independence of the MMMs**

**Thm.** Fix $n$. \exists g s.t.
\[
\mathbb{Q}[e_1, e_2, \ldots] \longrightarrow H^*(\text{MCG}(S_g)); \mathbb{Q}
\]
is injective up to degree $2n$ \quad (in fact $g=3n$).

i.e. \[
\mathbb{Q}[e_1, e_2, \ldots] \longrightarrow H^*(\text{MCG}(S_g))
\]

**Pf.** Choose $g_1, \ldots, g_n$ s.t. $e_i \in \text{MCG}(S_{g_i})$ is non-zero \quad $i=1, \ldots, n$.

(i.e. do our bundle construction for surfaces with boundary)

Choose $d_j$ s.t. \quad $j \cdot d_j \geq n$, \quad set \quad $g = \sum d_j g_j$

\[
\mathbb{L} : \text{MCG}(S_{d_1})^{d_1} \times \cdots \times \text{MCG}(S_{d_n})^{d_n} \longrightarrow \text{MCG}(S_g)
\]

**Fact:** \[
\mathbb{L}^*(e_i) = \sum_{j=1}^{n} \mathbb{P}_j(e_i) \quad \mathbb{P}_j = \text{proj to j}^{\text{th}} \text{ factor}
\]

(the point is that the euler classes live in separate subbundles).

Now just apply the Künneth formula. The image of any polynomial of deg $\leq 2n$ will have one term in the direct sum of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \otimes 1 \otimes \cdots \otimes 1$

which is $\neq 0$ by construction.
Computing $H_2$.

- First show $e_1$ generates a $\mathbb{Z}$ in $H^2(MCG(S_g))$ for $g \geq 3$.
- Then use Hopf formula to show $H^2(MCG(S_g))$ is a quotient of $\mathbb{Z}$ for $g \geq 4$ and of $\mathbb{Z} \oplus \mathbb{Z}_2$ for $g = 3$.
- Remains to show $H^2(MCG(S_3)) = \mathbb{Z} \oplus \mathbb{Z}_2$.

There is: $1 \rightarrow I(S_3) \rightarrow MCG(S_3) \rightarrow Sp_6(\mathbb{Z}) \rightarrow 1 \rightarrow$ 5-term sequence:

$$H_2(MCG(S_3)) \rightarrow H_2(Sp_6(\mathbb{Z})) \rightarrow H_1(I(S_3)) \rightarrow H_1(MCG(S_3)) \rightarrow H_1(Sp_6(\mathbb{Z}))$$

But: $H_1(MCG(S_3)) = 0$.

$H_2(Sp_6(\mathbb{Z})) = \mathbb{Z} \oplus \mathbb{Z}_2$ Stein '75.

Remains: $H_1(I(S_3))_{Sp_6(\mathbb{Z})} \cong [I(S_3)]/[MCG(S_3), I(S_3)] \cong 1$. Johnson '79

If: $a \circlearrowright \oplus b \circlearrowleft \oplus c \circlearrowright \oplus d \circlearrowleft$ and choose $h \in MCG(S_3)$ s.t. $h(b) = a$.

$$\ln I/[MCG,I]: [T_b, l] = h [T_b, l] h^{-1} \quad \text{since } [T_b, l] \in I(S_3)$$

$$= [hTbh^{-1}, h^{-1}lh^{-1}]$$

$$= [T_a, l [l^{-1}, h]]$$

$$= [T_a, l] l [T_a, [l^{-1}, h]] l^{-1}$$

$$= 1 \quad \text{since } T_a \leftrightarrow l$$

and $l \leftrightarrow h$ in $Sp$.

(since $[l^{-1}, h] \in I$).

Benson-Cohen: $H_2(MCG(S_3))$ consists of 2, 3, 5-torsion only.
Madsen-Weiss Theorem

We know \( \partial[E_1, E_2, \ldots] \to H^*(\text{MCG}(S^2)) \)

Want to show this is surjective.

Will do this by relating \( H^*(\text{MCG}(S^2)) \) to a "familiar" space.

\[ S_{\infty} = \ldots \]

\( \mathcal{G}_{S^2} = \) space of subsurfaces of \((0, g] \times \mathbb{R}^n\) diffeo to \( S^2 \) and

that agree on \( \partial S^2 \) with a fixed embedding of \( S_{\infty} \).

\[ = K(\text{MCG}(S^2), 1) \]

\( \mathcal{G}_{S^2} \to \mathcal{G}_{S^2+1} \to \mathcal{G}_{S_{\infty}} = \bigcup \mathcal{G}_{S^2} \)

Hovee stability \( \Rightarrow H_i(\mathcal{G}_{S_{\infty}}) = \lim g H_i(\mathcal{G}_{S^2}) = \lim g H_i(\text{MCG}(S^2)) \)

\[ \text{AG}_{n,d} = \text{affine Grassmannian of } \mathbb{A}^d \text{-planes in } \mathbb{R}^n \]

\[ \cong G_{n,d} \] since affine plane determined by plane thru 0 & 1 vector

\[ \text{AG}_{n,d}^+ = \text{1-pt comp} \]

\[ \cong \text{Thom space for } G_{n,d} \text{ when } n < \infty. \]

Theorem. \( H_*(\mathcal{G}_{S_{\infty}}) \cong H_*(\bigotimes_2^\infty \text{AG}_{n,2}) \) basept \( \infty \)

In general, the RO-cohomology of a loop space is a tensor product of a polynomial algebra on even-dim gens and an exterior alg. on odd-dim gens (assuming the loop space is path conn and has f.g. \( \mathbb{Z} \)-homology in each dim).
Scanning Map

Take some point in $G_n, s_g$ :

With a small lens we either see an almost-flat 2-plane or $\varnothing$. If we identify the lens with $\mathbb{R}^n$, get a pt in $AG^+_n, 2$ (slope is same as in lens but position of plane given by lens $\rightarrow \mathbb{R}^n$).

Near $\infty$, lens sees $\varnothing \mapsto$

$$S^n = \mathbb{R}^n \cup \{ \infty \} \rightarrow AG^+_n, 2$$

i.e. a point in $\Omega^n AG^+_n, 2$

As we move in $G_n, s_g$ can vary the size of the lens continuously.

As we let $n$ increase, have:

$$G_n, s_g \leftarrow G_{n+1}, s_g \downarrow \downarrow$$

$$\Omega^n AG^+_n, 2 \rightarrow \Omega^{n+1} AG^+_n, 2$$

where bottom row obtained by applying $\Omega^n$ to the map $AG^+_n, 2 \rightarrow \Omega AG^+_n, 2$ obtained by translating a plane from $-\infty$ to $\infty$ in $n+1$ st coord.

Taking limit over $n$:

$$G \rightarrow \Omega^\infty AG^+_n, 2$$

"Scanning map"

Note that the target does not depend on $g$, which is why we should expect to consider some limit over $g$ in order to get an isomorphism.
Fix $d$ (for us $d=2$)

$C^d$ = space of all smooth, oriented $d$-dim submanifolds of $\mathbb{R}^n$

that are properly embedded (maybe disconnected, open, empty).

Topology: pts are close if they are close in $C^d$ top. on a large ball

Note $C^d$ is path conn: radial expansion from a pt not on the

manifold gives a path to the empty manifold.

Prop. $C^d = AG^{+,d}$

Pf. Want to rescale from $O$, but this is not continuous since

we can push a manifold off $O$, changing image from

nonempty plane to empty plane.

Fix: For $M \in C^d$ choose tubular nbd $N = N(M)$ continuously.

If $O \notin N$, rescale as above.

If $O \in N$, rescale in tangent dir from $1 \rightarrow \infty$ as before

in normal dir $1 \rightarrow \lambda$ where

$\lambda = 1$ near $O$-sec, $\lambda = \infty$ near frontier.

This takes $AG^{+d}$ to itself.

Filter $C^d$ by

$C^{d,0} \leq C^{d,1} \leq \ldots \leq C^{d,n} = C^d$

where $C^{d,k}$ = subspace of $C^d$ consisting of manifolds lying in $\mathbb{R}^k \times (0,1)^{n-k}$

i.e. manifolds that extend to $\infty$ in only $k$ directions.

There is: $C^{d,k} \rightarrow \infty C^{d,k+1}$ by translating from $-\infty$ to $\infty$ in

$(k+1)$st coord.
Putting these together:

\[ C^n,0 \rightarrow L^n C^n,1 \rightarrow L^2 C^n,2 \rightarrow \cdots \rightarrow L^n C^n \]

The composition takes a compact manifold and translates it to \( \infty \) in all directions. (can think of this as scanning with an \( \infty \)ly large lens); shrinking the lens gives a homotopy to the original scanning map).

Would like: \( C^n,k \rightarrow L^2 C^n,k+1 \) is a homotopy equivalence.

Easier: \( k > 0 \) case. works for any \( d > 0 \).

Harder: \( k = 0 \) case. when \( d = 2 \), works after passing to limits where \( n, q \rightarrow \infty \). uses group completion theorem.

only get a homotopy equivalence:

\[ H_\ast(C^\infty) \cong H_\ast(L^2 C^\infty,1) \]

So the main thread for the MW Thm is:

\[
\begin{align*}
H_\ast(C^\infty) & \cong H_\ast(L^2 C^\infty,1) \\
& \cong \lim H_\ast(L^2 C^n) \\
& \cong \lim H_\ast(L^2 C^n) \\
& \cong \lim H_\ast(L^2 \mathbb{A}G^n) \\
& \cong H_\ast(L^2 \mathbb{A}G^\infty) \\
& \cong H_\ast(L^2 \mathbb{A}G^\infty) \\
\end{align*}
\]

harder delooping

easier delooping

above Prop.
DELOOPING - THE EASIER CASE

Want: $C^{n,k} \simeq \Omega C^{n,k+1}$ \quad $k > 0$.
Root map: $C^{n,k} \simeq M^{n,k} \simeq \Omega BM^{n,k} \simeq \Omega C^{n,k+1}$

Step 1. $M^{n,k} = \{ (M,a) \in C^n \times [0,\infty) : M \in \mathbb{R}^k \times (0,a) \times \mathbb{S}(0,1)^{n-k-1} \}$

This is a monoid version of $C^{n,k}$, analogous to the Moore loop space, which is a monoid version of $\Omega X$.

The map $C^{n,k} \to M^{n,k}$

$M \mapsto (M,1)$

is a homotopy equivalence.

Step 2. $M^{n,k} \simeq \Omega BM^{n,k}$

A topological monoid $M$ has a classifying space $BM$.
Construction is analogous to group case: $p$-simplices $\leftrightarrow (m_1,...,m_p)$ faces obtained by dropping $m_i, m_p$ & multiplying $m_i m_{i+1}$

There is a space of $p$-simplices with topology from $\coprod_p \Delta^p \times M^p$ and face identifications.

There is a map $M \to \Omega BM$

$m \mapsto (m)$

General fact: This is a hom. eq. when $\Pi_0 M$ is a group with mult. coming from mult. in $M$.

So we want: $\Pi_0 M^{n,k}$ is a group.
Prop. \( \pi_0 C^{n,k} = \begin{cases} \{0\} & k > d \\ \Omega_{d-k,n-k}^{S^0} & k \leq d \end{cases} \)

\( \uparrow \) cobordism group of closed, oriented \((d-k)\)-manifolds
in \( \mathbb{R}^{n-k} \).

**Pf.** A point of \( C^{n,k} \) is a \( d \)-manifold \( M \subseteq \mathbb{R}^n \)
with \( p : M \rightarrow \mathbb{R}^k \) proper.
Can perturb \( M \) s.t. \( p \) is transverse to \( 0 \in \mathbb{R}^k \).

\( k > d : p(M) \) misses \( 0 \). Expand radially from \( 0 \) in \( \mathbb{R}^k \) to get path to empty manifold.

\( k \leq d : p^{-1}(0) = M \cap \{0\} \times \mathbb{R}^{n-k} = M_0 \rightarrow [M_0] \in \Omega_{d-k,n-k}^{S^0} \)
\( \mapsto \varphi : \pi_0 C^{n,k} \rightarrow \Omega_{d-k,n-k}^{S^0} \)
\( [M] \mapsto [M_0] \)

This is a homomorphism since both group ops are disjoint union.
and surjective since \([\mathbb{R}^k \times M_0] \rightarrow [M_0]\)
Remains: \( \varphi \) injective.

First we claim any \( M \) is path conn to \( \mathbb{R}^k \times M_0 \) (first make \( M \) agree with \( \mathbb{R}^k \times M_0 \) on a nbhd of \( M_0 \), then expand radially)
Now if \( \varphi([M]) = [M_0] = \varphi([M']) \) equals \( \varphi([M']) = [M'_0] \)
can assume \( M = \mathbb{R}^k \times M_0 \) , \( M' = \mathbb{R}^k \times M'_0 \) and \( M \sim M'_0 \)
Build a manifold:

\( \mathbb{R}^k \times M_0 \) cobordism \( \sim \) \( \mathbb{R}^k \times M'_0 \)

Translating right gives path to \( \mathbb{R}^k \times M_0 \),
and left gives path to \( \mathbb{R}^k \times M'_0 \) so \( [M] = [M'] \) in \( \pi_0 C^{n,k} \).
STEP 3. $BM^{n,k} = C_0^{n,k+1}$

We will define a natural map $\nu : BM^{n,k} \rightarrow C_0^{n,k+1}$

A point in $BM^{n,k}$ is given by $(m_1, \ldots, m_p) \in (M^{n,k})^p$, $(w_0, \ldots, w_p)$

A stupid map (ignoring the $W_i$) is:

$$(m_1, \ldots, m_p) \mapsto m_1 m_2 \ldots m_p = \bigcup M_i \text{ where } M_i \text{ is a manifold with } (k+1)\text{st coord in } [a_i - 1, a_i]$$

This map is not continuous upon passage to faces:

1. When $w_0$ or $w_p \rightarrow 0$, $M_0$ or $M_p$ suddenly deleted.
2. When $w_0 \rightarrow 0$ $m_2 \ldots m_p$ suddenly shifts by $a_1 - a_0$ in $(k+1)\text{st coord}$

Can easily address 2: translate in $(k+1)\text{st coord}$ so

barycenter $b = \sum w_i a_i$ equals 0.

Idea for 1: truncate $M_0, M_p$ a little at a time.

precisely: $a_0^+ = \max \{a_0, b\} \quad b^+ = \sum w_i a_i$  \quad "upper & lower barycenters"

$a_0^- = \min \{a_0, b\} \quad b^- = \sum w_i a_i$

$\nu(M_1 \ldots m_p)$ obtained by stretching $\mathbb{R}^k \times (b^-, b^+) \times \mathbb{R}^{n-k-1}$

to $\mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}$

Need to check $\nu$ is $\cong$ on $\Pi_\phi \forall \phi$. 