NO BOUNDARIES
LIGHTNING TALKS
FRIDAY SESSION
Ivanov’s Metaconjecture

Tara Brendle
Dan Margalit

No Boundaries: Groups in Algebra, Geometry, and Topology
University of Chicago
October 27, 2017
Automorphisms of the Curve Complex

Theorem (Ivanov). \( \text{Aut } C(S_g) = \text{MCG}(S_g) \)

Application. \( \text{Aut } \text{MCG}(S_g) = \text{MCG}(S_g) \)
Rigidity for Complexes

- Systole Complex
  Schmutz-Schaller
- Nonseparating Curve Complex
  Irmak
- Pants Complex
  Margalit
- Complex of Separating Curves
  Brendle-Margalit
- Complex of Domains
  McCarthy-Papadopoulos
- Torelli Geometry
  Farb-Ivanov
- Arc Complex
  Irmak
- Arc and Curve Complex
  Irmak-Korkmaz
- Ideal Triangulation Graph
  Korkmaz
- Strongly sep. curve complex
  Bowditch
- Asymptotic Pants Complex
  Fossas-Nguyen
- Hole-bounding Curves and Pairs Complex
  Irmak-Ivanov-McCarthy
- Complex of Shirts and Straightjackets
  Bridson-Pettet-Souto
Rigidity for Groups

Mapping Class Group
Ivanov

Torelli Group
Farb-Ivanov

Johnson Kernel
Brendle-Margalit

Terms of Johnson Filtration
Bridson-Pettet-Souto
Ivanov’s Metaconjecture

Any object naturally associated to a surface $S$ and having a sufficiently rich structure has $\text{MCG}(S)$ as its group of automorphisms.
Rigidity for Groups

- Mapping Class Group
  - Ivanov

- Torelli Group
  - Farb-Ivanov

- Johnson Kernel
  - Brendle-Margalit

- Terms of Johnson Filtration
  - Bridson-Pettet-Souto

- Other Normal Subgroups?

- Dahmani-Guirardel-Osin examples
Main Theorem

If $N \triangleleft \text{MCG}(S_g)$ has an element with small support then:

$$\text{Aut } N = \text{MCG}(S_g).$$
Normal Subgroups of MCG

\[
\text{Aut} \gg \text{MCG} \quad \text{Aut} = \text{MCG}
\]

- Infinitely generated RAAGs
- Terms of Johnson filtration, Magnus filtration, etc.
Normal Subgroups of MCG

Aut >> MCG

Infinitely generated RAAGs

Aut = MCG

Terms of Johnson filtration, Magnus filtration, etc.
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The Primitive Torsion Problem

Khalid Bou-Rabee
Joint with Patrick W. Hooper
The Primitive Torsion Problem

• Let $F_r$ be the free group of rank $r$. A *primitive element* is an element that is part of a basis for $F_r$.

• Let $P_k$ be the group generated by $k$th powers of all primitive elements in $F_r$.

**The Primitive Torsion Problem**: When is $F_r/P_k$ finite? Finitely presented? Solvable? Nilpotent?

• Similar questions for other groups may be asked...
Known results

• **Theorem** (Thomas Koberda and Ramanujan Santharoubane, 2015) For some \( k \geq 10 \), the group \( F_r/P_k \) is infinite.

• **Theorem** (Andrew Putman and Justin Malestein, 2017) Same result. Different proof.

• **Theorem** (Patrick W. Hooper and Bou-Rabee, 2017) The group \( F_2/P_k \) is finite if and only if \( k = 1,2,3 \). **Moreover**, \( F_2/P_4 \) is virtually nilpotent (we construct an explicit integral representation), and \( F_2/P_k \) is finitely presented for \( k = 1,2,3,4,5 \).
The Farey triangulation:
Normal generators for $F_2/P_2$

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Generator of $P_2$</th>
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<tbody>
<tr>
<td>$\infty$</td>
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Normal generators for $F_2/P_3$

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<td>2</td>
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Normal generators for $F_2/P_5$

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<td>$(ab^{-1}ab^{-2})^5$</td>
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New notion

• A representation of $F_2$ is *characteristic* if for any automorphism $\psi$ of $F_2$, there is an automorphism $\Psi$ of $GL(n, \mathbb{C})$ so that $\Psi \circ \rho \circ \psi^{-1}(g) = \rho(g)$ for all $g \in F_2$.

• We say $\rho: F_2 \to GL(n, \mathbb{C})$ is an *oriented characteristic representation* if:
  
  • For each $\psi \in Aut_+(F_2)$ there is an $M \in GL(n, \mathbb{C})$ so that $M \rho \circ \psi^{-1}(g)M^{-1} = \rho(g)$ for all $g \in F_2$.
  
  • For each $\psi \in Aut_-(F_2)$ there is an $M \in GL(n, \mathbb{C})$ so that $M \cdot \rho \circ \psi^{-1}(g) \cdot M^{-1} = \rho(g)$ for all $g \in F_2$. 
Improvement scheme

• Assume $\rho: F_2 \to GL(n, \mathbb{C})$ is an oriented characteristic representation factoring through $G_k$. We produce an oriented characteristic representation $\tilde{\rho}: F_2 \to GL(n + m, \mathbb{C})$ factoring through $G_k$ (hopefully with $m > 0$) so that there is a short exact sequence of the form $1 \to \mathbb{Z}^d \to \tilde{\rho}(F_2) \to \rho(F_2) \to 1$ where $d \geq 0$ is the rank of the abelian image $\tilde{\rho}(\ker \rho)$ (hopefully $d > 0$).

• Using this scheme we get an explicit faithful representation for $F_2 / P_4$ and infinite representations for $F_2 / P_k$ for $k \geq 4$.

• What will this scheme give us for $F_2 / P_5$? We are working on it.
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Algebraic Characterizations in the Mapping Class Group

Victoria Akin
An Example

The Point-Pushing Subgroup

\[ 1 \to P(S_g) \to \text{Mod}(S_g, \ast) \to \text{Mod}(S_g) \to 1 \]
An Example

Algebraic Characterization

- Abstractly isomorphic to $\pi_1(S_g)$
- Normal in $\text{Mod}(S_g)$
An Example

(Ivanov-McCarthy) $\text{Out}(\text{Mod}^\pm(S_g, *)) \cong 1$
An Example

(Ivanov-McCarthy) $\text{Out}(\text{Mod}^{\pm}(S_g, *)) \cong 1$

- Burnside:
  If a centerless group $G$ is characteristic in $\text{Aut}(G)$, then $\text{Aut}(\text{Aut}(G)) \cong \text{Aut}(G)$. That is, $\text{Out}(\text{Aut}(G)) \cong 1$. 
An Example

(Ivanov-McCarthy) $\text{Out}(\text{Mod}^{\pm}(S_{g,*})) \cong 1$

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- Dehn-Nielsen-Baer:
  $\text{Aut}(\pi_1(S_g)) \cong \text{Mod}^{\pm}(S_{g,*})$. 
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- **Burnside:**
  If a centerless group $G$ is characteristic in $\text{Aut}(G)$, then $\text{Aut}(\text{Aut}(G)) \cong \text{Aut}(G)$. That is, $\text{Out}(\text{Aut}(G)) \cong 1$.

- **Dehn-Nielsen-Baer:**
  $\text{Aut}(\pi_1(S_g)) \cong \text{Mod}^\pm(S_g, *)$.

- **Uniqueness of Point-Pushing:**
  $\text{Out}(\text{Aut}(\pi_1(S_g))) \cong \text{Out}(\text{Mod}^\pm(S_g, *)) \cong 1$. 

In General

For $H < G$ geometrically/topologically defined, can we find a purely algebraic characterization?
In General

For $H < G$ geometrically/topologically defined, can we find a purely algebraic characterization?

- Braid group?

$$1 \rightarrow \pi_1(\text{Conf}_n(S_g)) \rightarrow \text{Mod}(S_{g,n}) \rightarrow \text{Mod}(\Sigma_g) \rightarrow 1$$

- Disk pushing?
- Handle pushing?
In General

What other normal/non-normal subgroups are unique?
In General

What other normal/non-normal subgroups are unique?

- (with D. Margalit) Torelli? Johnson Kernel? Higher terms in the Johnson Series?
Thank you
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Representation stability for Finitely Generate Arrangements

No Boundaries
Oct 2017

Nir Gadish
Linear subspace arrangements

A collection $\bigcup_{i=1}^{n} L_i \subset \mathbb{C}^d$

linear subspaces
**Linear subspace arrangements**

A collection \( \bigcup_{i=1}^{n} L_i \subset \mathbb{C}^d \)

Determines \( M_A = \mathbb{C}^d \setminus \bigcup_{i=1}^{n} L_i \)
Linear subspace arrangements

A collection \( \bigcup_{i=1}^{n} L_i \subset \mathbb{C}^d \) is linear subspaces

Determines \( M_A = \mathbb{C}^d \setminus \bigcup_{i=1}^{n} L_i \)

Fundamental problem: compute \( H^*(M_A) \).
Linear subspace arrangements

A collection \( \bigcup_{i=1}^{n} L_i \subset \mathbb{C}^d \)

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Arno'ld, Orlik-Solomon, Goresky-MacPherson...
Linear subspace arrangements

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Fundamental problem: compute \( H^*(M_A) \).

Arno'ld, Orlik-Solomon, Goresky-MacPherson...
(and Farb!)
Examples
Examples

1) Configurations: $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{z_i = z_j\}$

"the braid arrangement".
Examples

1) Configurations: \[ \mathbb{C}^n \setminus \bigcup \bigcup_{i \neq j} \{ z_i = z_j \} \]
"the braid arrangement".

2) Rational maps: \[ \mathbb{C}^n \times \mathbb{C}^n \setminus \bigcup \bigcup_{i,j} \{ z_i = p_j \} \]
Examples

1) **Configurations:**  \( C^n \setminus \bigcup_{i \neq j} \{ z_i = z_j \} \)

"the braid arrangement".

2) **Rational maps:**  \( C^n \times C^n \setminus \bigcup_{i,j} \{ z_i = p_j \} \)

3) **Type B:**  \( C^n \setminus \bigcup \{ z_i = \pm z_j \} \)
Examples

1) Configurations: \( \mathbb{C}^n \setminus \bigcup_{i \neq j} \{ z_i = z_j \} \)
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Notice: (a) group actions!
Examples

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Notice: (a) group actions!
(b) come in families!
Examples

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Notice: (a) group actions!
(b) come in families!

Goal: Understand $H^*(M_A)$ in this context.
Mechanism: C-subspace arrangements
Mechanism: C-subspace arrangements

Family = functor!
Mechanism: C-subspace arrangements

Family = functor!

e.g. \textbf{FI} = \textbf{Finite set and Injective functions}.

\[
\begin{align*}
\{1\} & \rightarrow \{1, 2\} \rightarrow \{1, 2, 3\} \rightarrow \ldots \rightarrow \{1, \ldots, n\} \rightarrow \ldots \\
S_1 & \rightarrow S_2 \rightarrow S_3 \rightarrow \ldots \rightarrow S_n
\end{align*}
\]
Mechanism: C-subspace arrangements

Family = functor!

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S_1 & \rightarrow & S_2 & \rightarrow & S_3 & \rightarrow & \cdots & \rightarrow & S_n \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}_1 & \rightarrow \mathcal{A}_2 & \rightarrow \mathcal{A}_3 & \rightarrow & \cdots & \rightarrow & \mathcal{A}_n & \rightarrow & \cdots \\
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Mechanism: \(C\)-subspace arrangements

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& \downarrow \quad \downarrow \quad \downarrow \\
S_1 & \quad \quad S_2 \quad \quad S_3 \\
& \quad \downarrow \\
& \quad \quad S_n \\
A_1 & \rightarrow A_2 \rightarrow A_3 \rightarrow \ldots \rightarrow A_n \rightarrow \ldots \\
& \downarrow \quad \downarrow \quad \downarrow \\
& \quad \quad S_1 \quad \quad S_2 \quad \quad S_3 \\
& \quad \downarrow \\
& \quad \quad S_n
\end{align*}
\]

One object! e.g. braid arrangements.

\[
\mathbb{C}^* \setminus \bigcup_{i \neq j} \{z_i = z_j\}
\]
Mechanism: C-subspace arrangements

Family = functor!

e.g. $\textbf{FI} = \text{Finite set and Injective functions.}$

$\{1\} \rightarrow \{1, 2\} \rightarrow \{1, 2, 3\} \rightarrow \ldots \rightarrow \{1, \ldots, n\} \rightarrow \ldots$

$\circ \quad \circ \quad \circ \quad \circ \quad \circ$
$S_1 \quad S_2 \quad S_3 \quad \ldots \quad S_n$

$\mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_3 \rightarrow \ldots \rightarrow \mathcal{A}_n \rightarrow \ldots$

$\circ \quad \circ \quad \circ \quad \circ \quad \circ$
$S_1 \quad S_2 \quad S_3 \quad \ldots \quad S_n$

One object! e.g. braid arrangements.

$\mathbb{C}^* \setminus \bigcup_{i \neq j} \{z_i = z_j\}$ only "one equation" (?)
Representation stability
Representation stability

Applying cohomology:

\[
\begin{array}{c}
\text{Arr} \\
\downarrow H^i \\
\text{FI} \longrightarrow \text{Vect}
\end{array}
\]
Representation stability

Applying cohomology:

\[
\begin{array}{ccc}
\text{FI} & \longrightarrow & \text{Vect} \\
\downarrow & & \downarrow \\
\text{Arr} & \longrightarrow & A.
\end{array}
\]

get an \textbf{FI}-module - 

\[ [n] \mapsto H^i(M_{A_n}). \]
**Representation stability**

Applying cohomology:

\[
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\downarrow & & \downarrow \text{H}^i \\
\text{FI} & \rightarrow & \text{Vect}
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\]

get an **FI**-module -

\[ [n] \mapsto \text{H}^i(M_{A^n}). \]

**Theorem [G]:** the **C**-module \( H^*(M_A) \) of a finitely generated **C**-arrangement exhibits representation stability.
Representation stability

Applying cohomology:

\[
\begin{array}{ccc}
\text{Arr} & \downarrow \text{H}^i \\
A. & \downarrow \\
\text{FI} \rightarrow \text{Vect}
\end{array}
\]

get an \textbf{FI}-module - \quad \quad [n] \leftrightarrow H^i(M_{A_n}).

**Theorem [G]:** the \textbf{C}-module $H^\ast(M_A)$ of a finitely generated \textbf{C}-arrangement exhibits representation stability.

(a) Polynomial dimensions.
**Representation stability**

Applying cohomology:

\[
\begin{array}{ccc}
\text{FI} & \rightarrow & \text{Vect} \\
\downarrow & & \downarrow \\
\text{Arr} & \rightarrow & H^i
\end{array}
\]

get an \( \text{FI} \)-module -

\[
[n] \mapsto H^i(M_{\mathcal{A}_n}).
\]

**Theorem [G]:** the \( \mathcal{C} \)-module \( H^*(M_{\mathcal{A}}) \) of a finitely generated \( \mathcal{C} \)-arrangement exhibits representation stability.

(a) Polynomial dimensions.
(b) Polynomial characters.
**Representation stability**

Applying cohomology:

\[ \text{FI} \rightarrow \text{Vect} \]

get an \( \text{FI} \)-module -

\[ [n] \rightarrow H^i(M_{A^n}). \]

**Theorem [G]:** the \( \mathcal{C} \)-module \( H^*(M_A) \) of a finitely generated \( \mathcal{C} \)-arrangement exhibits representation stability.

(a) Polynomial dimensions.
(b) Polynomial characters.
(c) Inductive description.
Concrete consequences
Concrete consequences

1. Configuration space

\[ \chi_{H^2(PConf^n(\mathbb{C}))} = 3\binom{X_1}{1} + \binom{X_1}{2}X_2 - \binom{X_2}{2} - X_4 + 2\binom{X_1}{3} - X_3 \]
Concrete consequences

1. Configuration space

$$\chi_{H^2(PConf^n(\mathbb{C}))} = 3\binom{X_1}{1} + \binom{X_1}{2}X_2 - \binom{X_2}{2} - X_4 + 2\binom{X_1}{3} - X_3$$

$$X_k(\sigma) = \# k\text{-cycles in } \sigma$$
Concrete consequences

1. Configuration space

\[ \chi_{H^2(PConf^n(C))} = 3 \binom{X_1}{1} + \binom{X_1}{2} X_2 - \binom{X_2}{2} - X_4 + 2 \binom{X_1}{3} - X_3 \]

\[ X_k(\sigma) = \# k\text{-cycles in } \sigma \]

2. Rational maps

\[ \dim H^3(Prat^n(C)) = 12 \binom{n}{2} \binom{n}{3} + 2n \binom{n}{3} + 3 \binom{n}{2} \binom{n}{2} \]
Concrete consequences

1. Configuration space

\[ \chi_{H^2(PConf^n(\mathbb{C}))} = 3 \binom{X_1}{1} + \binom{X_1}{2} X_2 - \binom{X_2}{2} - X_4 + 2 \binom{X_1}{3} - X_3 \]

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Applications

• SET-free sets [Harman].
Concrete consequences

1. Configuration space

\[ \chi_{H^2(PConf^n(\mathbb{C}))} = 3 \binom{X_1}{1} + \binom{X_1}{2} X_2 - \binom{X_2}{2} - X_4 + 2 \binom{X_1}{3} - X_3 \]

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Applications

- SET-free sets [Harman].
- Arithmetic statistics of rational maps.

via Étale cohomology.
Thank you!

Any questions?
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Bounding the cohomology of configuration spaces and rationality of Poincaré series

Kevin Casto
Configuration spaces

- \( \text{PConf}_n(M) = \{(m_i) \in M^n \mid m_i \neq m_j\} \)

- \( \text{Conf}_n(M) = \text{PConf}_n(M)/S_n \)

- So \( H^i(\text{PConf}_n(M); \mathbb{Q}) \) is an \( S_n \)-representation, and
  \( H^i(\text{PConf}_n(M))^{S_n} = H^i(\text{Conf}_n(M)) \)

\[ \in \text{Conf}_4(\Sigma_2) \]
Representation stability

- Recall that irreps of $S_n$ are parameterized by partitions: $\{S^\lambda | \lambda \vdash n\}$

- If $m \geq n + \lambda_1$, can extend to $\lambda[m] = (m - n, \lambda_1, \ldots, \lambda_k) \vdash m$

- Given $\{V_n\}$ with $V_n$ an $S_n$-rep, satisfies representation stability [CF] if $\langle V_n, S^{\lambda[n]} \rangle_{S_n}$ is eventually constant

- Church [Ch] proved $H^i(\text{PConf}_n(M))$ satisfies repr. stability for a “nice” manifold $M$.

- Taking the trivial rep, this means $H^i(\text{Conf}_n(M))$ satisfies homological stability
What about varying $i$?

- In applications, need to bound $\langle H^i(P\text{Conf}_n(X)), S^\lambda[n] \rangle$ as $i$ varies

- *A priori*, rep stability doesn’t help, since that’s only about each fixed $i$

- **Theorem ([Ca]).** For $M$ “nice”,

  $$|\langle H^i(P\text{Conf}_n(M)), S^\lambda[n] \rangle| \leq P(i)$$

  where $P(i)$ is a polynomial independent of $n$
Poincaré series rationality

- Put

\[ F_{M,\lambda}(x) = \sum_{i \geq 0} \langle H^i(\text{PConf}(M)), S^{\lambda[n]} \rangle t^i \]

- Basic fact: if a power series is rational and has poles at roots of unity, its coefficients are a quasipolynomial

- Means there are poly’s \( p_0, \ldots, p_{d-1} \) s.t. \( a_i = p_i \mod d(i) \), so \( a_i \) bounded by a polynomial

- **Question:** Is \( F_{M,\lambda}(x) \) always rational with poles roots of unity (for \( M \) nice)?
Partial results

- Question inspired by W. Chen [Che] – using work of [KL], showed answer is “yes” for $M = \mathbb{C}$ (explicit formula)

- Farb-Wolfson-Wood [FWW] prove answer is yes for the trivial rep $(\lambda = \emptyset)$ if $M$ is a conn. open submanifold of $\mathbb{R}^{2r}$

- In this case $(\lambda = \emptyset)$ we are just looking at power series of stable Betti numbers of $\text{Conf}_n(X)$

- Orlik-Solomon [OS] says that

$$H^*(\text{PConf}_n(\mathbb{C})) = \Lambda^* \langle e_{ij} \rangle / (e_{ij}e_{jk} + e_{jk}e_{ik} + e_{ik}e_{ij})$$

If we don’t quotient by ideal, calculations suggest analogous question for exterior algebra fails!
References


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Discretizing group actions (Vigolo, ’16)

- $\Gamma$ f.g. group
- $M$ closed Riem. manifold
- $\Gamma \curvearrowright M$ (bi-Lipschitz)

$\Rightarrow$

Family of graphs $(X_t)_{t>0}$
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$\Gamma \curvearrowright M \xrightarrow{\text{Mesh}} (X_t)_{t>0}$

Action $\Gamma \curvearrowright M$

Graphs $X_t$

Vertices: Regions $R_i$
Edges: $sR_i \cap R_j \neq \emptyset$.
Discretizing group actions (Vigolo, ’16)

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$\leadsto$ Family of graphs $(X_t)_{t>0}$

Action $\Gamma \curvearrowright M$

Mesh $< t^{-1}$

Roe’s Warped Cone

Assembles all $X_t$

$\leadsto \mathcal{C}(\Gamma \curvearrowright M)$. 
Dynamics and coarse geometry

Dynamics of $\Gamma \curvearrowright M$

Coarse geometry of graphs $(X_t)_t$
Or Warped Cone $C(\Gamma \curvearrowright M)$

Theorem (Vigolo, '16)
Spectral gap for $\Gamma \curvearrowright M = \Rightarrow (X_n)_n$ expander.

Sawicki $\iff$ Subgroups of compact Lie groups $\mapsto$ Spectral gap
Margulis, Sullivan, Drinfeld, Gamburd–Jakobson–Sarnak, Bourgain–Gamburd ($\times 2$), Benoist-De Saxc´ e, ...

From now on: $M = G$ compact semisimple Lie
$\Gamma \subseteq G$ dense, fin. pres.
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Theorems

Coarse geometry of cones $\iff$ Dynamics of $\Gamma \bowtie M$

Theorem (De Laat–Vigolo, Sawicki, '17)
Warped cones are $\text{QI} \iff$ Groups are Stably $\text{QI}$

$C(\Gamma \bowtie M) \cong \text{QI} C(\Lambda \bowtie N) \iff \Gamma \times \mathbb{R} \text{dim } M \cong \text{QI} \Lambda \times \mathbb{R} \text{dim } N$

Does the QI type of the cone capture any of the action?

Theorem (Fisher–Nguyen–vL, '17)
Warped cones are $\text{QI} \iff$ actions are commensurable

Similar result for graphs $\iff$

Theorem (Fisher–Nguyen–vL, '17)
There exist continua of QI disjoint expanders.
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Warped cones are QI \[ \implies \] Groups are \textit{Stably} QI

\[ \mathcal{C}(\Gamma \acts M) \simeq_{QI} \mathcal{C}(\Lambda \acts N) \implies \Gamma \times \mathbb{R}^{\dim M} \simeq_{QI} \Lambda \times \mathbb{R}^{\dim N}. \]

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