Rigidity implies geometricity for surface group representations

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Rigidity

Γ discrete group (e.g. \( \pi_1(\Sigma_g) = \Gamma_g \)), \( G \) topological group
Study representations \( \rho : \Gamma \to G \).

*think:* \( G \) linear (rep. theory) or \( G = \text{Homeo}(M), \text{Diff}(M) \) (dynamics)

**Definition:** \( \rho : \Gamma \to G \) is rigid if “only trivial deformations”
\[
\rho \in \text{Hom}(\Gamma, G)/G \text{ is an isolated point.}
\]

**Problem:** quotient space typically not Hausdorff
e.g. \( \text{Hom}(\mathbb{Z}, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C}) \leftrightarrow \text{trace except } \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Solution:** Define “character space”
\( X(\Gamma, G) := \text{largest Hausdorff quotient of } \text{Hom}(\Gamma, G)/G \)
for \( \text{SL}(n, \mathbb{C}) \) this *is* characters; for \( G \) complex, reductive Lie group, it is GIT quotient

Change definition: Rigid means isolated point in \( X(\Gamma, G) \).
Rigidity from geometry

**Mostow rigidity (Calabi):** \( \Gamma = \pi_1(M^n) \) hyperbolic manifold
\( \Gamma \to \text{SO}(n,1) \) embedding as cocompact lattice is rigid in \( X(\Gamma, \text{SO}(n,1)) \)

*Analog in non-linear setting?*

**Definition:** \( \rho : \Gamma \to \text{Homeo}(M) \) is *geometric* if factors through
\( \Gamma \hookrightarrow G \twoheadrightarrow \text{Homeo}(M) \)
cocompact lattice
transitive Lie group

**Example 1.** \( \pi_1(\Sigma_g) \to \text{PSL}(2,\mathbb{R}) \to \text{Homeo}(S^1) \)

**Theorem (Matsumoto ’87)**
The example above is *rigid* in \( X(\pi_1(\Sigma_g), \text{Homeo}(S^1)) \).
Geometric reps to $\text{Homeo}(S^1)$

**Fact:** Connected, transitive Lie groups in $\text{Homeo}(S^1)$ are
- $\text{SO}(2)$
- finite cyclic extensions of $\text{PSL}(2, \mathbb{R})$

$\mathbb{Z}/k\mathbb{Z} \to G \to \text{PSL}(2, \mathbb{R})$

**Cor.:** can describe all geometric actions of $\pi_1(\Sigma_g) = \Gamma_g$ on $S^1$.
(lifts of Fuchsian actions)

**Theorem (Mann, 2014)**
If $\rho : \Gamma_g \to \text{Homeo}(S^1)$ is geometric, then it is rigid.

**Theorem (Mann–Wolff, 2017)**
Converse: if $\rho \in X(\Gamma_g, \text{Homeo}_+(S^1))$ is rigid, then it is geometric.
Plan:

1. What is $X(\Gamma_g, \text{Homeo}_+(S^1))$?

2. Idea of proof for rigid $\Rightarrow$ geometric.
What is $X(\Gamma_g, \text{Homeo}_+(S^1))$?

- Space of flat (foliated), topological $S^1$ bundles over $\Sigma_g$
- Points are semi-conjugacy classes of actions
- Parametrized by rotation numbers of elements. analog of trace coordinates for $X(\Gamma, \text{SL}(2, \mathbb{R}))$
- Topologically... complicated

Not known:
- Finitely many connected components?
- How different from $X(\Gamma_g, \text{Diff}_+(S^1))$? (see work of J. Bowden)
Proof ideas for “Rigid ⇒ Geometric”

Dynamical lemma: \( \rho \) rigid \( \Rightarrow \) \( \rho(\gamma) \) has rational rotation number for every simple closed curve \( \gamma \).

Key tool: \textit{Bending deformations} works in \( \text{Hom}(\Gamma_g, G) \) for any \( G \)

\[ \Gamma_g = A \ast \langle c \rangle \ast B \]

Bending \( \rho \) along \( c \): take \( c_t \) commuting with \( \rho(c) \).
Define \( \rho_t = c_t \rho c_t^{-1} \) on \( B \),
\[ \rho_t = \rho \text{ on } A. \]

\[ \Gamma_g = F \ast \langle a \rangle \]

Bending \( \rho \) along \( a \): similar, define \( \rho_t(b) = a_t \rho(b) \).
if \( a_1 = \rho(a) \), like Dehn twist

Headaches: • based curves. • centralizers. • 1-parameter subgroups.
Proof ideas for “Rigid ⇒ Geometric”

Main idea: \( \rho(\gamma) \) has periodic points (lemma), so take bending \( \rho_t \) and study movement of periodic points of \( \rho_t(\gamma) \).

\( \rho \) rigid ⇒ combinatorial structure of \( \text{Per}(\rho_t(a)), \text{Per}(\rho_t(b)) \)

“won’t change” e.g. having common point, cyclic order of points

From this, “reconstruct” the structure of geom. rep.
Suppose $\rho(a)$ and $\rho(b)$ have hyperbolic dynamics.
Baby version of main idea

Suppose $\rho(a)$ and $\rho(b)$ have hyperbolic dynamics:

Claim: $\rho$ rigid $\Rightarrow$ axes cross.
Suppose $\rho(a)$ and $\rho(b)$ have hyperbolic dynamics:

**Claim:** $\rho$ rigid $\Rightarrow$ axes cross. “reconstruct topology of $\Sigma_g$”

**Proof:** Suppose $\rho_t(b) = a_t\rho(b)$

$\rho_t(a) = a$

Picture: axis of $a^{-N}\rho(b)$ for $N \gg 0$:

repelling point near $\rho(b)^{-1}(a_+)$

deformation gives non-conjugate picture, contradiction \(\square\)
This line of argument “works” if $|\text{Per}(\rho(a))| < \infty$.

- “axes” of SCC’s “intersect” only when (based) curves do.
- w/ combinatorial technique of Matsumoto (2015), get geometricity.

Much work to arrive at deformation so that $|\text{Per}(\rho(a))| < \infty$, build machinery to modify and track combinatorics of periodic sets.

Many open questions remain...
Thanks!