Let \( S_g \) denote a closed, connected, orientable surface of genus \( g \), and let \( \text{Mod}(S_g) \) denote its mapping class group, that is, the group of homotopy classes of orientation preserving homeomorphisms of \( S_g \).

**Fact.** If \( g \geq 2 \), then every Dehn twist in \( \text{Mod}(S_g) \) has a nontrivial root.

It follows from the classification of elements in \( \text{Mod}(S_1) \cong \text{SL}(2,\mathbb{Z}) \) that Dehn twists are primitive in the mapping class group of the torus.

For Dehn twists about separating curves, the fact is well-known: if \( c \) is a separating curve then a square root of the Dehn twist \( T_c \) is obtained by twisting one side of \( c \) through an angle of \( \pi \). In the case of nonseparating curves, the issue is more subtle. We give two equivalent constructions of roots below.

**Geometric construction.** Fix \( g \geq 2 \). Let \( P \) be a regular \((4g-2)\)-gon. Glue opposite sides to obtain a surface \( T \cong S_{g-1} \). The rotation of \( P \) about its center through angle \( 2\pi g/(2g-1) \) induces a periodic map \( f \) of \( T \). Notice that \( f \) fixes the points \( x,y \in T \) that are the images of the vertices of \( P \). Let \( T' \) be the surface obtained from \( T \) by removing small open disks centered at \( x \) and \( y \). Define \( f' = f|T' \).

Let \( A \) and \( B \) be annular neighborhoods of the boundary components of \( T' \). Modify \( f' \) by an isotopy supported in \( A \cup B \) so that

- \( f'|\partial T' \) is the identity,
- \( f'|A \) is a \((g/(2g-1))\)-left Dehn twist, and
- \( f'|B \) is a \(((g-1)/(2g-1))\)-right Dehn twist.

Identify the two components of \( \partial T' \) to obtain a surface \( S \cong S_g \) and let \( h : S \to S \) be the induced map. Then \( h^{2g-1} \) is a left Dehn twist along the gluing curve, which is nonseparating.

**Algebraic construction.** Let \( c_1,\ldots, c_k \) be curves in \( S_g \) where \( c_i \) intersects \( c_{i+1} \) once for each \( i \), and all other pairs of curves are disjoint. If \( k \) is odd, then a regular neighborhood of \( \bigcup c_i \) has two boundary components, say, \( d_1 \) and \( d_2 \), and we have a relation in \( \text{Mod}(S_g) \) as follows:

\[
(T_{c_1}^2 T_{c_2} \cdots T_{c_k})^k = T_{d_1} T_{d_2}.
\]

This relation comes from the Artin group of type \( B_n \), in particular, the factorization of the central element in terms of standard generators; it also follows from the \( D_{2p} \) case of [2, Proposition 2.12(i)]. In the case \( k = 2g-1 \), the curves \( d_1 \) and \( d_2 \) are isotopic nonseparating curves; call this isotopy class \( d \). Using the fact that \( T_d \) commutes with each \( T_{c_i} \), we see that

\[
[(T_{c_1}^2 T_{c_2} \cdots T_{c_{2g-1}})^{1-g}T_d]^{2g-1} = T_d.
\]
Roots of half-twists. Let $S_{0,2g+2}$ be the sphere with $2g + 2$ punctures (or cone points) and let $d$ be a curve in $S_{0,2g+2}$ with 2 punctures on one side and $2g$ on the other. On the side of $d$ with 2 punctures, we perform a left half-twist, and on the other side we perform a $(g-1)/(2g-1)$–right Dehn twist by arranging the punctures so that one puncture is in the middle, and the other punctures rotate around this central puncture. The $(2g-1)^{st}$ power of the composition is a left half-twist about $d$. Thus, we have roots of half-twists in $\text{Mod}(S_{0,2g+2})$ for $g \geq 2$. There is a 2-fold orbifold covering $S_g \rightarrow S_{0,2g+2}$ where the relation from our algebraic construction above descends to this relation in $\text{Mod}(S_{0,2g+2})$ [1, Theorem 1 plus Corollary 7.1]. A slight generalization of this construction gives roots of half-twists in any $\text{Mod}(S_n)$ with $n \geq 5$.

Roots of elementary matrices. If we consider the map $\text{Mod}(S_g) \rightarrow \text{Sp}(2g, \mathbb{Z})$ given by the action of $\text{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z})$, we also see that elementary matrices in $\text{Sp}(2g, \mathbb{Z})$ have roots; for instance, we have

$$
\begin{pmatrix}
1&0&0&1\\
0&1&0&0\\
0&1&-1&0\\
1&0&0&1
\end{pmatrix} ^3 = \begin{pmatrix}
1&1&0&0\\
0&0&1&0\\
0&0&0&1
\end{pmatrix} .
$$

By stabilizing, we obtain cube roots of elementary matrices in $\text{Sp}(2g, \mathbb{Z})$ for $g \geq 2$.

Roots of Nielsen transformations. Let $F_n$ denote the free group generated by $x_1, \ldots, x_n$, let $\text{Aut}(F_n)$ denote the group of automorphisms of $F_n$, and assume $n \geq 2$. A Nielsen transformation is an element of $\text{Aut}(F_n)$ conjugate to the one given by $x_1 \mapsto x_1 x_2$ and $x_k \mapsto x_k$ for $2 \leq k \leq n$. The following automorphism is the square root of a Nielsen transformation in $\text{Aut}(F_n)$ for $n \geq 3$.

$$
x_1 \mapsto x_1 x_3 \\
x_2 \mapsto x_3^{-1} x_2 x_3 \\
x_3 \mapsto x_3^{-1} x_2
$$

Passing to quotients, this gives a square root of a Nielsen transformation in $\text{Out}(F_n)$ and, multiplying by $-\text{Id}$, a square root of an elementary matrix in $\text{SL}(n, \mathbb{Z})$, $n \geq 3$. Also, our roots of Dehn twists in $\text{Mod}(S)$ can be modified to work for punctured surfaces, thus giving “geometric” roots of Nielsen transformations in $\text{Out}(F_n)$.

Other roots. If $f \in \text{Mod}(S_g)$ is a root of a Dehn twist $T_d$, then $f$ commutes with $T_d$. Since $fT_c f^{-1} = T_{f(c)}$ for any curve $c$, we see that $f$ fixes $d$. In the complement of $d$, the class $f$ must be periodic. This line of reasoning translates to $\text{GL}(n, \mathbb{Z})$ and $\text{Aut}(F_n)$: roots correspond to torsion elements in $\text{GL}(n-1, \mathbb{Z})$ and $\text{Aut}(F_{n-1})$, respectively. In all cases, one can show that the degree of the root is equal to the order of the torsion element.

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References


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