Linear Algebra, Midterm 2

April 5, 2009

1. Suppose that \( V \) is a subspace of \( \mathbb{R}^4 \) generated by the three vectors \((1,1,-1,1), (4,-1,-2,3)\) and \((4,-7,0,-1)\). Using the Gram-Schmidt process find three orthogonal vectors that span \( V \).

2. Find the determinant of the following matrix:
\[
\begin{pmatrix}
1 & 2 & 1 & 1 \\
2 & 6 & 1 & 6 \\
3 & 6 & 6 & 1 \\
4 & 8 & 7 & 3
\end{pmatrix}.
\]

3. There does not exist a vector \( x = [x_1 \ x_2]^t \) such that
\[
\begin{pmatrix}
1 & 2 \\
-1 & 1 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.
\]

However, one can find a “closest” vector \( x \) (or “least squares” vector) by projecting the vector \([1 \ 2 \ 2]^t\) onto the column space of the \(3 \times 2\) matrix on the left-hand-side. Find this projection, and find the corresponding \( x_1 \) and \( x_2 \) which gives the “closest” vector.

4. Find the eigenvalues and eigenvectors of the following matrix:
\[
A = \begin{pmatrix}
-7 & 2 \\
-15 & 4
\end{pmatrix}.
\]

Then, find a matrix \( S \) and a diagonal matrix \( \Lambda \), such that
\[
A = SAS^{-1}.
\]

5. Give an example of a \(2 \times 2\) matrix \( A \) such that
a. \( A^8 = I \), but \( A^1, A^2, ..., A^7 \neq I \).

b. The matrix \( A \) is singular, while the sum of its eigenvalues \( \lambda_1 + \lambda_2 = 4 \).