Practice Math 110 Final

Instructions: Work all of problems 1 through 5, and work any 5 of problems 10 through 16.

1. Let

\[ A = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 3 & 2 \\ 6 & 6 & 5 \end{bmatrix}. \]

a. Use Gauss elimination to reduce \( A \) to an upper triangular matrix (row reduced form).

b. Define what is meant by an elementary matrix.

c. Write down the elementary matrices corresponding to the row operations (interchange two rows, multiply a row by a scalar, and add one row to another).

2.

a. Define what is meant by the inverse of a matrix.

b. Find the inverse of the matrix \( A \) in question 1.

3.

a. Define what is meant by a symmetric matrix.

b. Show that if \( A \) and \( B \) are \( n \times n \) matrices, and \( A \) is symmetric, then \( A^2 \) and \( B'AB \) are symmetric (\( B' \) means the transpose of \( B \)).

c. Define what is meant by a permutation matrix \( P \). Show that there exists a power \( n \geq 1 \) such that if \( P \) is a permutation matrix, then \( P^n = I \), where \( I \) is the identity matrix.

4.

a. Let \( V \) be a vector space. Define what is meant by a basis for \( V \). Also define the dimension of \( V \).

b. Let \( A \) be an \( m \times n \) matrix, and let \( U \) be the normal form of \( A \). Define the row space of \( A \), the column space of \( A \), the rank of \( A \), the nullity of \( A \), and the nullspace of \( A \).

5.

a. State the dimension theorem.

b. Given the linear map

\[ T : \mathbb{R}^3 \to \mathbb{R}^4, \]
which sends
\[(a, b, c) \rightarrow (a + b, 2b, 3a + b, c + b),\]
find the matrix representation for \(T\) with respect to the bases
\[(1, 0, 0), (0, 1, 1), (1, 0, 1)\]
for \(\mathbb{R}^3\), and
\[(1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 1, 0), (1, 0, 0, 1)\]
for \(\mathbb{R}^4\).
6. Suppose that \(A\) is an \(n \times n\) matrix, and that \(v\) is an eigenvector of \(A\) with eigenvalue \(\lambda\). If \(P(x)\) is any polynomial, show that
\[P(A)v = P(\lambda)v.\]
Note: The \(P(A)\) is a matrix, whereas the \(P(\lambda)\) is a scalar.
7. Assume the claim in problem 6 is true, suppose that \(A\) is an \(n \times n\) matrix, and that \(f(x)\) is its minimal polynomial, and that \(c(x)\) is its characteristic polynomial. Prove that if \(\lambda\) is a root of \(c(x)\), then it is also a root of \(f(x)\).
   Hint: If \(\lambda\) is root of \(c(x)\), then it is an eigenvalue, which must have a non-zero eigenvector \(v\). Since \(f(x)\) is the minimal polynomial, we must have \(f(A) = 0\) (that is, \(f(A)\) is the 0 matrix), and so
   \[f(A)v = 0.\]
What can you conclude, assuming problem 6.
8. Let \(A\) be an \(n \times n\) matrix with real entries. Prove that the subspace
\[\text{Span}(I_n, A, A^2, A^3, ...)\]
has dimension \(\leq n\) (even though the matrix \(A\) has \(n^2\) entries and lies in the vector space \(\text{Mat}_{n \times n}(\mathbb{R})\), which has dimension \(n^2\)).
9. Suppose that \(V\) is a vector space equipped with an inner product \(<\cdot, \cdot>\). Let \(v, w\) be two linearly independent vectors in \(V\). Prove that the vectors
\[v \text{ and } w - \lambda v, \text{ where } \lambda = \frac{<v, w>}{<v, v>}\]
are orthogonal. You are NOT allowed to say that this follows directly from Gram-Schmidt. The point of the problem here is to show that you know some things about inner products, as well as the definition of orthogonal.

10.  
   a. Prove that a matrix and its transpose have the same eigenvalues.
   b. Suppose $P$ is a transition matrix of a markov chain, whose entries $P_{i,j}$ give the probability that a “atom” transitions from state $j$ to state $i$, and note that the sum of the entries in each column is 1. Prove that $\lambda = 1$ is an eigenvalue for $A$ (Hint: Use part a.).

11. Let $N$ be the set of all nilpotent matrices in $\text{Mat}_{n \times n}(\mathbb{R})$. For $n \geq 2$, $N$ is NOT a subspace; for example,

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

are both nilpotent, but their sum $A + B$ is not nilpotent, so this proves that $N$ cannot be a subspace.

However, if $W$ is any subspace of $\text{Mat}_{n \times n}(\mathbb{R})$ such that every member of $W$ commutes with every other member of $W$ under matrix multiplication, then prove that

\[W \cap N \quad \text{is a subspace of} \quad \text{Mat}_{n \times n}(\mathbb{R}).\]

Hint: This problem is similar to the problem on your first exam, where you were asked to show that $A, B$ nilpotent implies $A + B$ nilpotent.

12. Suppose that $X, Y, Z$ are finite-dimensional vector spaces, and that $f : X \to Y$ and $g : Y \to Z$ are both linear maps. Prove that

\[\dim(\text{im}(g \circ f)) \leq \dim(\text{im}(f)).\]

Hint: In the composition $g \circ f$, the map $f$ first maps $X$ into $\text{im}(f)$, and then $g$ maps $\text{im}(f)$ into $\text{im}(g)$. So this last map is actually $g|_{\text{im}(f)}$ (that is, $g$ restricted to the image of $f$). Applying the dimension theorem to the map $g|_{\text{im}(f)}$ and the departure space $\text{im}(f)$, we get

\[\dim(\text{im}(f)) = \dim(\ker(g|_{\text{im}(f)})) + \dim(\text{im}(g|_{\text{im}(f)})).\]

Note that $\text{im}(g|_{\text{im}(f)}) = \text{im}(g \circ f)$. So,...
13. Suppose that $V$ is a finite-dimensional vector space, and that $T$ is a linear map from $V$ to $V$. Further, let $x$ be a vector in $V$, and let $k$ be the least integer such that

$$x, T(x), ..., T^{k-1}(x)$$

are all linearly independent.

Then, show that the subspace

$$S = \text{Span}(x, T(x), ..., T^{k-1}(x))$$

is $T$-invariant; that is, show that if $s \in S$, then $T(s) \in S$.

14. 
   a. Define what is meant by a determinental map.
   b. Show that if $A$ is an $n \times n$ matrix with real entries, and $D$ is a determinental map from $\text{Mat}_{n \times n}(\mathbb{R})$ to $\mathbb{R}$, then if $A$ has two identical columns, $D(A) = 0$.

15. Label the following as true or false:
   a. The rank of a matrix is equal to the number of its non-zero columns.
   b. Elementary row operations applied to a matrix preserve the rank of a matrix.
   c. If $A$ is an $m \times n$ matrix, where $m \leq n$, then the rank of $A$ is at least equal to $m$.
   d. Any system of $n$ linear equations in $n$ unknowns has at least one solution.
   e. Any system of $n$ linear equations in $n$ unknowns has at most one solution.
   f. Any polynomial of degree $n$ and leading coefficient $(-1)^n$ is the characteristic polynomial of some $n \times n$ matrix $A$.
   g. The characteristic polynomial of a matrix always has degree larger than the minimal polynomial.

16. Assume $A$ is an invertible $n \times n$ matrix with integer entries. Prove that the entries of $A^{-1}$ have integer entries if and only if the $\det(A) = \pm 1$.

   Hint: Look at the adjugate of $A$, and recall that

$$[\text{adj } A]_{i,j} = (-1)^{i+j}\det(A_{j,i}),$$
where $A_{j,i}$ is gotten by taking $A$ and removing the $j$th row and $i$th column. Also recall that

$$A(\text{adj } A) = \det(A)I_n.$$ 

How can you use this to prove the claim that

$$A^{-1} \text{ has integer entries } \implies \det(A) = \pm 1?$$