The two portfolios puzzle

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1 Introduction

First, I would like to say that I learned of this puzzle some years ago from Yang Wang, who is presently department head of Michigan State University mathematics, and was formerly professor here at Georgia Tech.

Here we consider a problem about the performance of two portfolios, where the one with the *higher* expected yearly returns turns out to be, somewhat paradoxically, the poorer performer in the long run:

Problem. Suppose one has two portfolios, which have the following performance characteristics:

• From year to year of investing with portfolio 1, there is a 60% chance of *increasing* ones money by 30%, while there is a 40% change of *decreasing* ones money by 20%.

• From year to year of investing with portfolio 2, there is a 70% chance of *increasing* ones money by 40%, while there is a 30% chance of *decreasing* ones money by 50%.

If one sticks with one portfolio or the other, reinvesting the money produced by each particular portfolio year after year (compounding ones money), which portfolio will do better in the long run?

Let us compute the expected returns of each portfolio: At the end of year 1, if one invests with portfolio 1, one expects that ones initial investment will be magnified by a factor of

$$(0.6)(1.3) + (0.4)(0.8) \;=\; 1.1$$

(i.e. the expected returns are 1.1 times as large as at the start of the year). And with portfolio 2, one expects that ones initial investment will be magnified by a factor of

$$(0.7)(1.4) + (0.3)(0.5) \;=\; 1.13.$$

So, just on the basis of "expected returns" it would seem that portfolio 2 is the better of the two. Is this right?

2 Solution

The answer, surprisingly, turns out to be 'no' – the first portfolio, even though it has a lower expected return, is the better performer in the long run.

Basically, what goes wrong is that the "most likely" outcome of investing with one portfolio or another is far different from the "expected" or "average" outcome. This happens because the rare events where, say, the second portfolio performs exceedingly well, more than make up for their low probability of occurrence, thereby skewing the expected returns far away from the "most probable" ones. This can perhaps be called a type of "fat-tailed" phenomena.

So what should one do? What one should do is compute the expected returns of the *logarithm* of the two portfolios, and then compare those. I don't want to get into the reasons for this, elementary though they may be. Suffice it to say that upon taking logs one can tranform the problem into one of looking at binomial distributions when many re-investments are made, and then those binomials can be analyzed by the normal distribution approximation.

The expected value of the logarithm of the multiplication factor at the end of year one for portfolio 1 is

$$(0.6)\ln(1.3) + (0.4)\ln(0.8) \;=\; 0.06816...$$

and for portfolio 2 it is

$$(0.7)\ln(1.4) + (0.3)\ln(0.5) \;=\; 0.02758...$$

Clearly, portfolio 1 is the victor!

3 Another way to look at it

Now let us modify the problem slightly, whereby from year to year one is able to invest in, say, 100 different packages of independent stocks, each and every one having the same performance characteristic, which we assume is either portfolio 1 or 2. So, say you have 1 million dollars to invest, then you split the investment up into 100 independent package investments of $10,000 each. Then, at the end of year one, you take the returns and divide them up amongst the independent package investments again, and continue this year after year.

If one is able to adopt such a "diversified" investment strategy, then it turns out that this time it is better to invest in portfolio 2.

I leave it to the reader to sort out why this is the case.