Math 4108 Study Sheet for Midterm 1

February 23, 2010

1. Know the book's definition of prime elements, irreducible elements of polynomial rings. Know how to show that if p is a prime element of a PID, and $p|ab$, then $p|a$ or $p|b$ (on the exam, I will tell you which version of the definition of "prime element" to use). Know how to compute gcd's in Euclidean Domains, in particular in $F[x]$, where F is a field.

2. Know the fact that fields have only trivial ideals. Know that if f is a non-trivial homomorphism out of a field, then f must be injective. Know the fact that if D is a domain, and I is a maximal ideals of D, then D/I is a field. Know the characteristic of a domain. Know the fact that every finite integral domain is a field.

3. Understand the proof that if D is a Euclidean Domain, then D is also a UFD; also, know the fact that this holds more generally for PID. In particular, know how to show $F[x]$ is a ED, where F is a field; so, $F[x]$ is a UFD. Know the obvious example showing the $Z[x]$ is not a PID (The example is I is the set of all polynomials with an even constant coefficient. Another way to describe I is that it is the set of all polynomials $f(x)$ where $f(0)$ is even.)

4. Know basic properties about the content of a polynomial, and understand how to use it to show that $Z[x]$ is a UFD. Know Gauss's Lemma and Eisenstein's criteria for irreducibility.

5. Know the fact that if F is a field, then $F[x]$ is a UFD. Know that if R is a UFD, then so is $R[x]$, and in fact, $R[x_1, ..., x_k]$.

6. Know what is meant by "extension field", "finite extension", and "degree of extension". Know what the notation $[K : F]$ means in the context of finite extensions. Know the "product rule" for towers of extensions: If K is a finite extension of F, and L is a finite extension of K, then $[L : F] = [K : F][L : K]$. Be able to compute the degree of various finite extensions.

7. Know what it means for an element $\alpha \in K$ to be "algebraic over F" (K is an extension of F). Know what is meant by an "algebraic extension". Are all algebraic extensions finite? Are all finite extensions algebraic? Know how to give examples and/or counterexamples. Know the transitivity relation for algebraic extensions, namely that if K is algebraic over F, and L is algebraic over K, then L is algebraic over F. It might be worthwhile to have some appreciation of how this is proved.

8. Know what "algebraic of degree n over F" means. Know the fact that if a, b are algebraic over F, then so are $a \pm m$, ab and a/b ($b \neq 0$). Know how to prove this fact.

9. Know what is meant by "algebraic number". Know that $e = 2.718281828...$ is not an algebraic number (i.e. it is not algebraic over the rationals; so, it is called 'transcendental').

10. Know what is meant by a "root of a polynomial $f(x) \in F[x]$". Know the "root-factor lemma", which says that if a is a root of f, then $(x - a)|f(x)$. Be able to use this to prove that if $f(x)$ has degree n, then it can have at most n roots, counting multiplicities.

11. Know how to construct an field extension of F in which a given polynomial $f(x)$ has a root. (Furthermore, the degree of this extension is at most the degree of f). Know how to use this to construct an extension L in which "f splits completely" (this extension will have degree at most $n!$ over F), meaning that if we think of $f(x) \in L[x]$, then $f(x)$ factors into linear polnomials.

12. Know what is meant by a "splitting field". Know that splitting fields are unique up to isomorphism. It is a good idea to have some appreciation of how this is proved.

13. Know what is meant by "constructible number". Know how to think of constructible numbers in terms of field extensions (and know how to come up with those extensions in the first place – you just basically consider the equations produced by intersecting circles and lines, and then solve them... the solutions lie in at worst a degree 2 extensions of the field generated by previous intersection points). Know how to use this to show that a 60 degree angle is not trisectible with straightedge and compass. Also, know how to show it is impossible to "double the cube". It is good to be aware of the following (originally conjectured by Gauss, proved in full by Pierre Wantzel in 1837): A regular n-gon is constructible if and only if $n = 2^m n_0$, where $m \geq 0$ and where n_0 is a product of distinct Fermat primes.