Constructing sequences leading to unbounded discrepancy

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Here is a vague suggestion for how to prove that completely multiplicative ± 1 functions $f(n)$ necessarily have unbounded discrepancy. I have no idea whether it will lead anywhere, but it is the *type* of idea that might get around some of these complicated "inverse theorems" (if $f(n)$ produces bounded discrepancy, then it correlates with a twisted Dirichlet character) that various approaches so far see to require.

The idea is to try to find integers

$$x_1,\ y_1,\ x_2, y_2,\ ...,\ y_k$$

such that we can produce a collection of sequences

$$\begin{gathered} n_{1,1},\ n_{1,2},\ ...,\ n_{1,k} \\ n_{2,1},\ n_{2,2},\ ...,\ n_{2,k-1} \\ n_{3,1},\ n_{3,2},\ ...,\ n_{3,k-2} \\ \vdots \\ n_{t,1},\ n_{t,2},\ ...,\ n_{t,k-t+1}, \end{gathered}$$

where all

$$n_{i,j+1} = n_{i,j} + 1,$$

and where

$$\begin{gathered} n_{1,1} = x_1 y_1,\ n_{1,2} = x_2 y_2,\ n_{1,3} = x_3 y_3, ... \\ n_{2,1} = x_1 y_2,\ n_{2,2} = x_2 y_3,\ n_{2,3} = x_3 y_4,\ ... \\ n_{3,1} = x_1 y_3,\ n_{3,2} = x_1 y_4,\ n_{3,3} = x_3 y_5,\ ... \\ \vdots \end{gathered}$$

(The ith row the x_i's have indices that are $i-1$ apart.)

Now it is probably easy to see that such a configuration is impossible – hopefully, though, some interesting (and useful) configurations *like this* are possible. One tool one could use to work out (or conjecture) legal possibilities is a "degrees of freedom" argument: it seems one would need that the number of x_i's and y_j's (which is $2k$ all told in our case) is at least as big as the number of equations (the equations are the $n_{i,j+1} = n_{i,j} + 1$) we want to satisfy.

I would think that this "degrees of freedom" idea, in combination with considering "local obstructions" (i.e. modulo small primes) and "global obstructions" (are the equations possible over the reals?), should pin down exactly what configurations are possible.

Now, let's assume that we could produce configurations like the above. It seems unlikely that sums along every row would have bounded discrepancy (since otherwise, the x_i's fail to correlate in a very strong way with many shifts of the y_i's). Ok, well, I suppose one could have the x_i's being all 1's and the y_j's alternating ± 1; so, maybe we would need that the set of values of the y_j's and the x_i's coincide. Another idea is to have each $n_{i,j}$ involve a product of three variables, instead of just 2 (this would give more degrees of freedom to play with).