CONWAY MUTATION AND ALTERNATING LINKS

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Abstract. Here are some follow-up questions that arose during my talk at the Tech Topology Conference on December 11, 2011. That talk reported on the results of my paper *Lattices, graphs, and Conway mutation* [Gre11]. I also wrote a shorter, more conversational accompaniment to it entitled *Conway mutation and alternating links* [Gre].

Given a pair of diagrams D and D' for a pair of links L and L', consider the following four statements:

(1) D and D' are mutants (as diagrams);
(2) L and L' are mutants (as links);
(3) the branched double-covers $\Sigma(L)$ and $\Sigma(L')$ are homeomorphic; and
(4) the d-invariant in Heegaard Floer homology of $\Sigma(L)$ and $\Sigma(L')$ are the same.

In general, we have (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). Our main result reads as follows.

Theorem 0.1. *For (connected, reduced)* alternating *diagrams D, D', (1)-(4) are* equivalent*.*

Thus, alternating links with homeomorphic branched double-covers are mutants, and the d-invariant is a *complete* invariant of the homeomorphism types of the spaces $\Sigma(L)$, L an alternating link.

This work leaves open several interesting avenues of inquiry. First, we remark that Theorem 0.1 was previously known to hold for two-bridge links [Bro60, Sch56, Rei35, Rus05]. We were directed to the present result by the following mantram, which we would like to advertise.

Mantram 0.2. *Generalize all questions and results about two-bridge links to alternating links.*

Recall that the branched double-cover of a two-bridge link is a lens space. Which properties of lens spaces persist for spaces of the type $\Sigma(L)$, where L is alternating? For example, is it possible to classify tight contact structures on the latter, by analogy to Honda's classification for lens spaces [Hon00, Thm.2.1]?

Second, how far does Theorem 0.1 extend beyond alternating links? There do exist pairs of non-mutant links whose branched double covers are homeomorphic, such as the torus knot $T(3, 7)$ and the pretzel knot $P(-2, 3, 7)$. It is a fascinating, wide-open problem to characterize pairs of links in S^3 with homeomorphic branched double covers; so fascinating, in fact, that it appears twice in Kirby's problem list [Kir10, Probs.1.22&3.25]. Mecchia-Zimmermann have some intriguing results along these lines; one asserts that if Y is hyperbolic, then there are at most nine non-isotopic links $L \subset S^3$ with $\Sigma(L) \cong Y$, and furthermore there exist examples showing that "nine" is optimal [MZ04]. However, we make the following conjecture.

Conjecture 0.3. *If a pair of links have homeomorphic branched double-covers, then either both are alternating or both are non-alternating.*

In support of Conjecture 0.3, Hodgson-Rubinstein showed that a lens space is the branched double-cover of a unique link in S^3, and this link is a two-bridge link [HR85, Cor.4.12]. (So Conjecture 0.3 is an instance of Mantram 0.2). Also, Menasco showed that a mutant of an alternating link is again alternating (which also establishes that (2) $\Longrightarrow$ (1) for alternating links above) [Men84, Proof of Thm.3(b)]. Perhaps more direct topological techniques, a là Bonahon [Bon83] or Hodgson-Rubinstein, will succeed in establishing Conjecture 0.3.

Third, in the proof of Theorem 0.1, it is essential that the spaces $\Sigma(L)$ bound a sharp 4-manifold with both orientations. This motivates a question.

Question 0.4. *Suppose that Y is a rational homology sphere, and Y bounds a sharp 4-manifold with both orientations. Does it follow that $Y \cong \Sigma(L)$ for some alternating link L?*

If this were the case, and in addition Conjecture 0.3 were true, then we would obtain a non-diagrammatic characterization of alternating links, albeit in very round-about terms. This is in the spirit of Ralph Fox's question, "What is an alternating knot?"

Fourth, and finally, we can read Theorem 0.1 as asserting that the d-invariant of $\Sigma(L)$ is a complete invariant of the mutation type of an alternating link L.

Question 0.5. *Is there a natural invariant coming from Floer homology that distinguishes the isotopy type of an alternating link?*

Here the hope would be to find a Floer-theoretic approach to the Menasco-Thistlethwaite theorem (formerly, the Tait flyping conjecture) [MT91].

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