Conway mutation and alternating links

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It is a basic operation for transforming one link $L \subset S^3$ into another $L' \subset S^3$.

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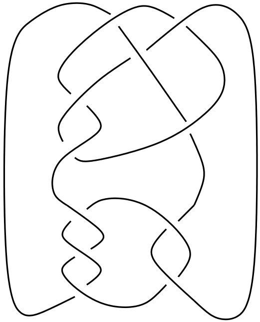

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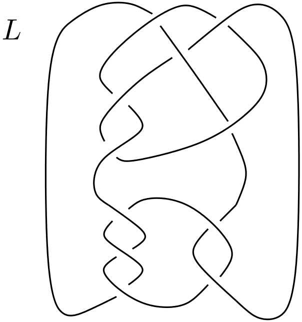

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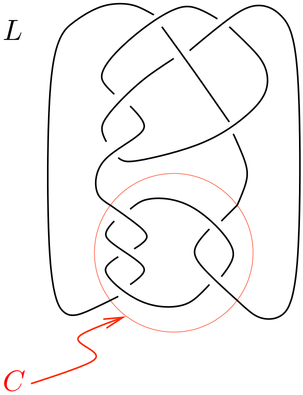

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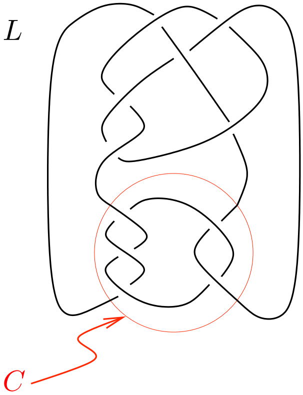

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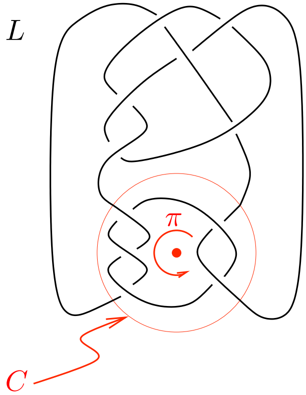

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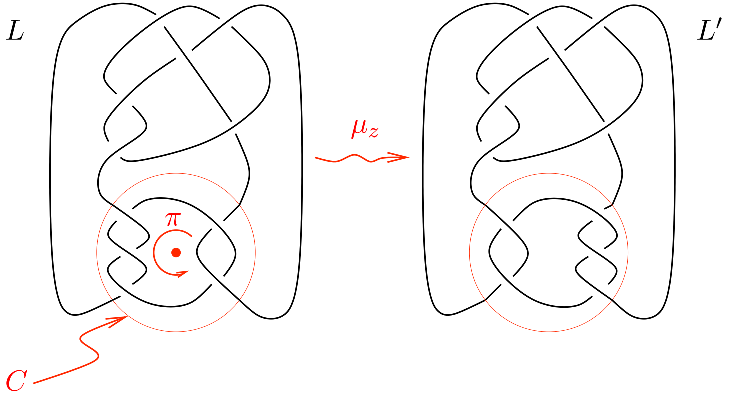

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Mutation preserves a number of well-known link invariants:

- the HOMFLY polynomial;
- the signature (for knots);
- hyperbolicity / hyperbolic volume (Ruberman);
- the odd Khovanov homology (Bloom);
- the homeomorphism type of $\Sigma(L)$, the double-cover of S^3 branched along L (Viro).

There do exist non-mutant links with homeomorphic branched double-covers, e.g. $P(-2,3,7)$ and $T(3,7)$ (distinct HOMFLY polynomials).

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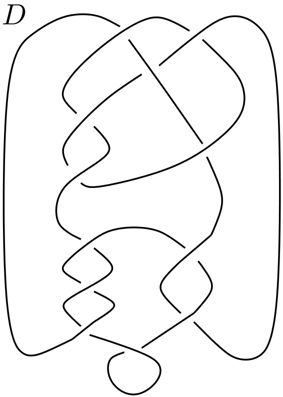
Note. $1. \implies 2. \implies 3. \implies 4.$ are immediate;
$2. \implies 1.$ follows from work of Menasco;
previously known to hold for two-bridge links (Reidemeister, Schubert).

An alternating diagram D gives rise to a *Tait graph* G:



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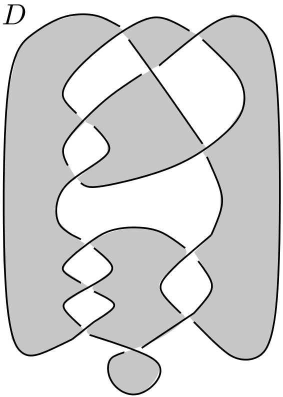

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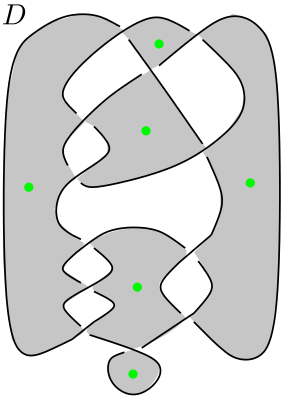

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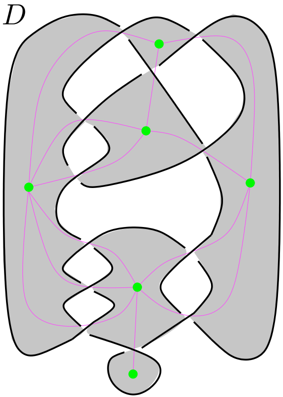

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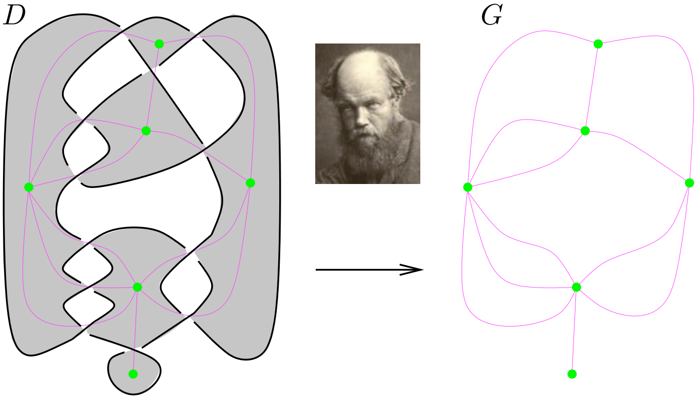

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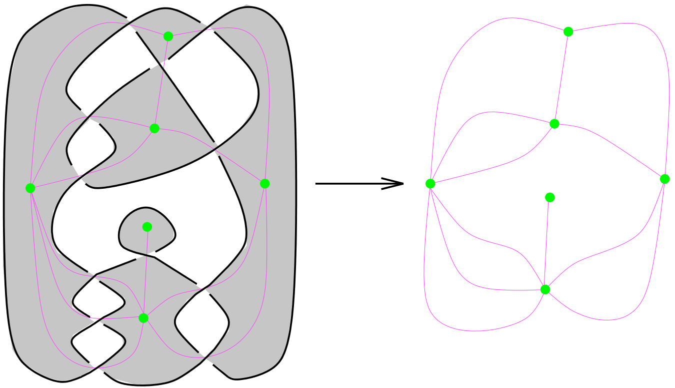

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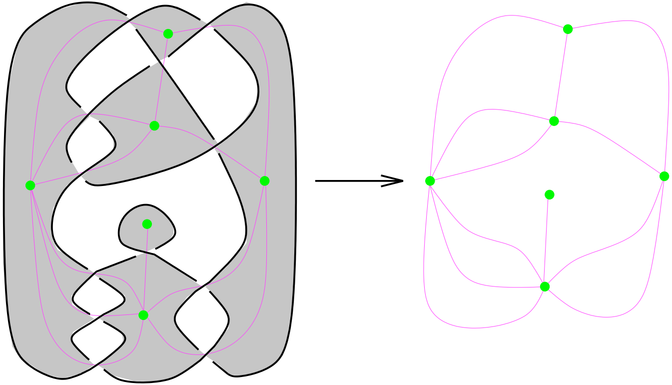

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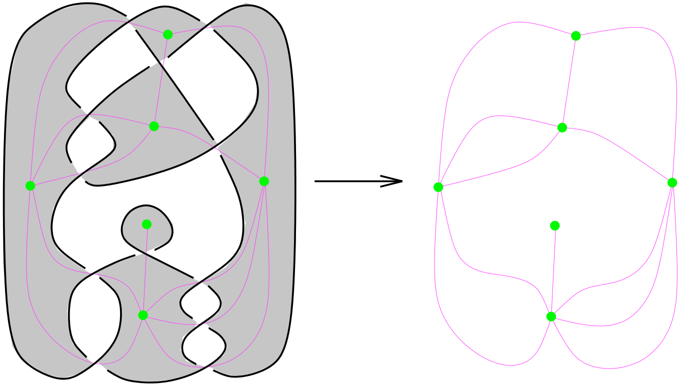

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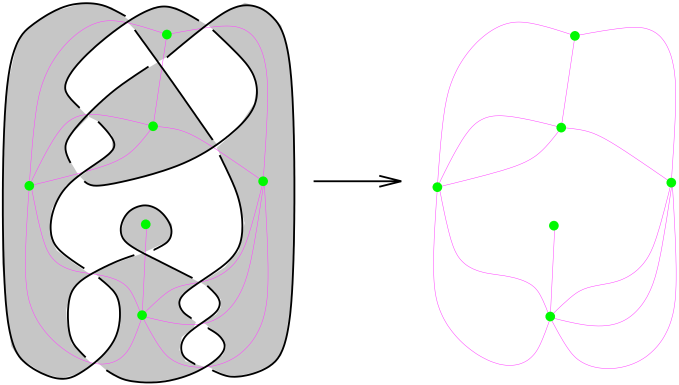

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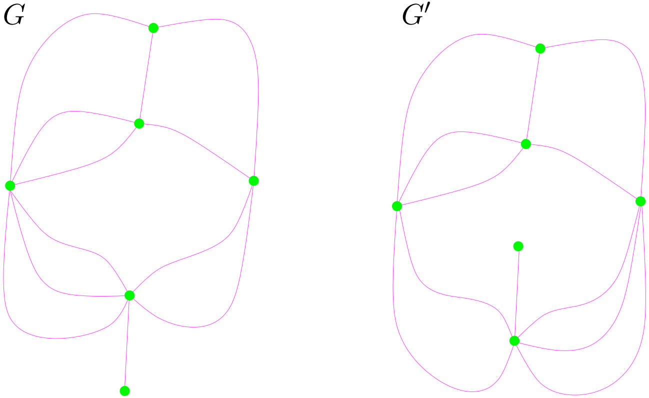

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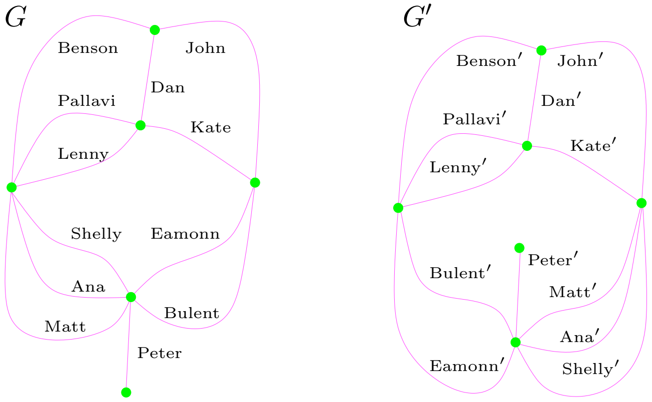
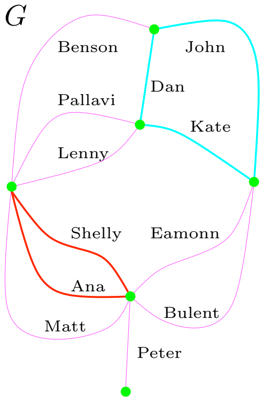
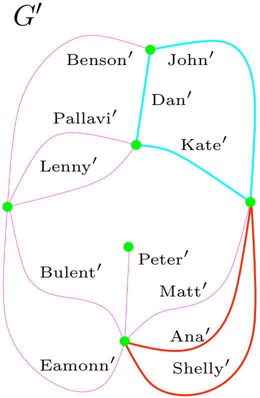
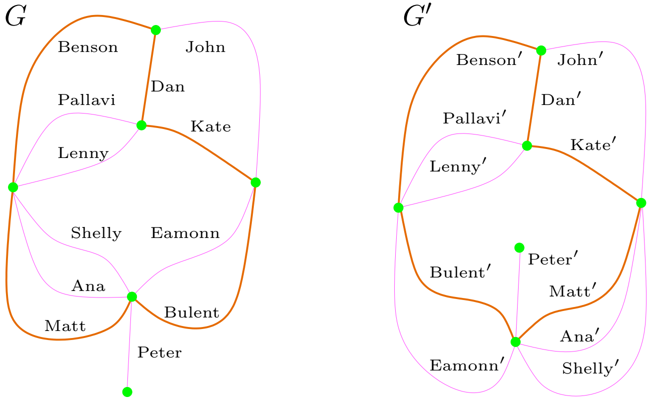


The highlighted cycles clearly get sent to one another since they are supported within the two individual “halves”.





This one is more interesting since it crosses the 2-vertex cutset.



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Proof sketch.

- Elementary mutations in diagrams effect flips and switches in the Tait graphs, and vice versa.
- A pair of plane drawings of a planar graph are related by flips (Whitney, Mohar-Thomassen).
- A pair of 2-isomorphic graphs are related by switches (Whitney).

□

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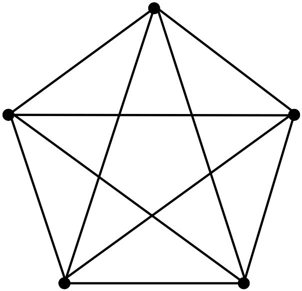

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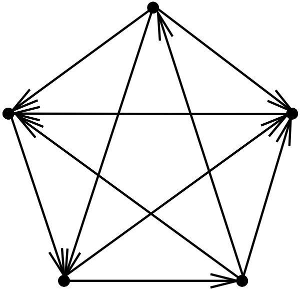

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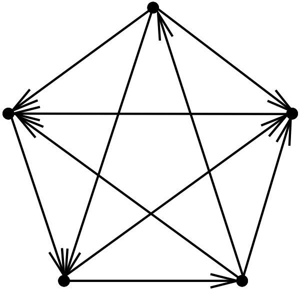

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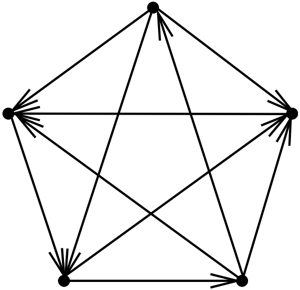

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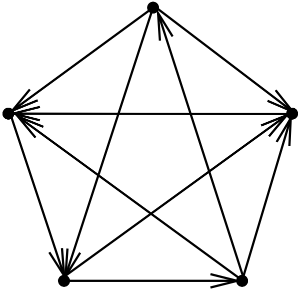

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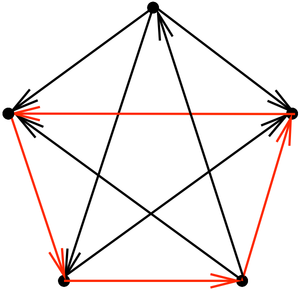

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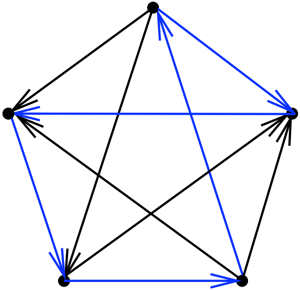

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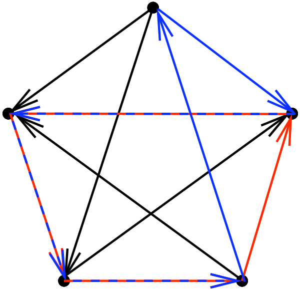
$$\langle x, y\rangle = 3$$



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The pair $(C(\Lambda), d)$ is the d-invariant of Λ. In short, it records the minimal norms of characteristic covectors in the various equivalence classes (mod 2Λ).

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Theorem 2 (Ozsváth-Szabó)

The space $\Sigma(L)$ is an L-space, and

$$(\mathrm{Spin}^c(\Sigma(L)), d) \xrightarrow{\sim} (C(\mathcal{F}(G)), -d).$$

Note. For an L-space Y, $(\mathrm{Spin}^c(Y), d)$ determines $\widehat{HF}(Y)$ as an absolutely graded, relatively spin^c-graded group.

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Note. 3. $\Longrightarrow$ 2. (Bacher-de la Harpe-Nagnibeda)
2. $\Longrightarrow$ 3. analogue of the Torelli theorem for a finite graph (Artamkin, Caporaso-Viviani, Su-Wagner)

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- By construction, its unique d-invariant vanishes.

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  - We obtain a composite map $f : E(G) \xrightarrow{\sim} \mathcal{B} \xrightarrow{\sim} E(G')$.
  - Since $\mathcal{F}(G)$ and $\mathcal{C}(G')$ are complementary within Λ, it follows that f is a 2-isomorphism.

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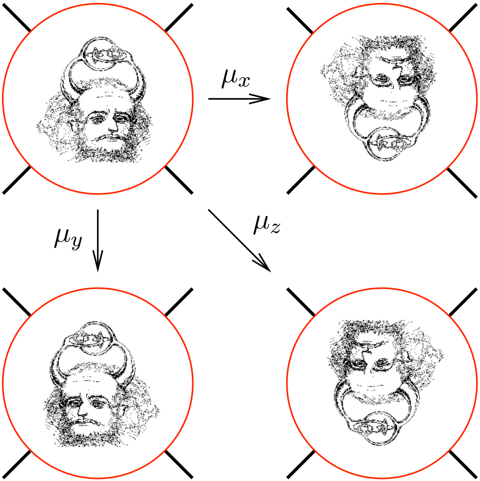
Question

Is there an analogous complete invariant of the isotopy type within the class of alternating links? Combining $d(\Sigma(L))$ and $\tau(\widetilde{L} \subset \Sigma(L))$, perhaps?

Cf. the Menasco-Thistlethwaite theorem: two reduced, alternating diagrams of a link differ by a sequence of flypes.

Mutation of Conway horned spheres:



Credits: Simon Fraser (Conway), wikipedia (Tait), IAS (Whitney), Mariana Cook (Elkies)