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From invariants of tangles to invariants of bordered 3-manifolds

Goal: motivate construction of bordered monopole Floer theory (joint w/ Jon Bloom)

Khovanov Homology: planar diagram D for L

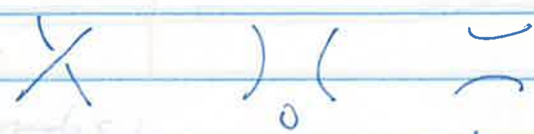
$\rightsquigarrow$ complex $(C(D), \partial)$

$Kh = H_*(\quad)$ a link inv't

link cobordism $F: L_0 \to L_1$

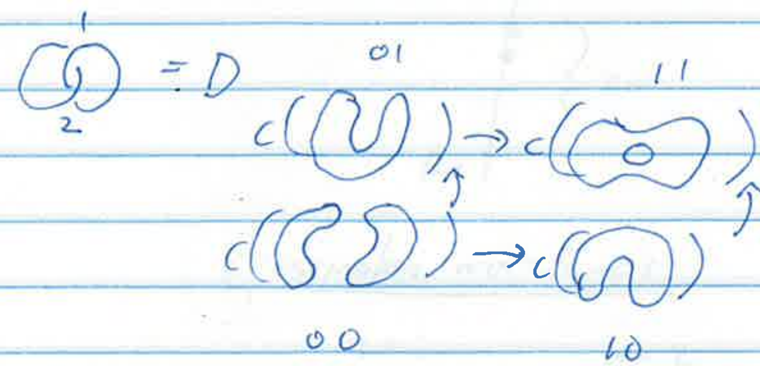
$\rightsquigarrow m(F): Kh(L_0) \to Kh(L_1)$

resolution



$C(D)$: complete resolution $\leftrightarrow I \in \{0,1\}^n \leftrightarrow D_I$

e.g.



$C(D) =$ sum of

$\partial =$ sum of "merge" and "split" maps associated to edges

D_I = union of circles $x_1 \ldots x_k$

$C(D_I) = \mathbb{F}_2[x_1, \ldots x_n] / x_i^2 = 0 \ \forall i$

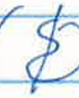
given a matching m, let $\bar{m}$ denote its reflection

$m_0 \cup \bar{m}_1$

$$H^n = \bigoplus_{\alpha,\beta} C(m_\alpha \cup \bar{m}_\beta)$$

↙ Khovanov cx for $m_\alpha \cup \bar{m}_\beta$

index matchings

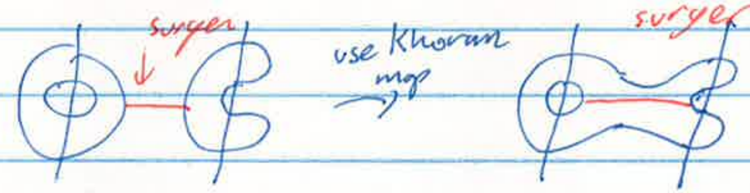
eg. $H^2 = C($  $) \oplus C($  $) \oplus C($  $) \oplus C($  $)$

4 2 2 4

<u>multiplications:</u>

$$C(m_\alpha \cup \bar{m}_\beta) \cup C(m_\beta \cup \bar{m}_\gamma) \to C(m_\alpha \cup \bar{m}_\beta)$$

eg.



$m_1 \cup \bar{m}_1$ $m_1 \cup \bar{m}_0$

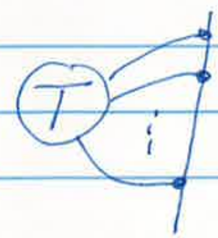
↙ use Khovanov map

so get composition of merge & split maps

<u>idempotents</u>: for each α, $1_\alpha \in C(m_\alpha \cup \overline{m}_\alpha)$

$$\forall x \in C(m_\alpha \cup \overline{m}_\beta)$$
$$1_\alpha \cdot x = x$$

for a tangle get a <u>module</u>

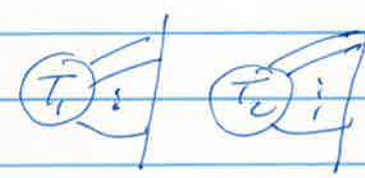


$$C(T) = \bigoplus_\alpha C(T \cup \overline{m}_\alpha) \qquad C(\overline{T}) = \bigoplus_\alpha C(m_\alpha \cup \overline{T})$$

multiplication
$$C(T \cup \overline{m}_\alpha) \otimes C(m_\alpha \cup \overline{m}_\beta) \to C(T \cup \overline{m}_\beta)$$

$C(T)$ invariant up to quasi-isomorphism

<u>Pairing</u>: Given tangles T_1, T_2



$$C(T_1 \cup \overline{T_2}) \underset{\substack{\uparrow \\ \text{quasi-isomorphic}}}{\simeq} C(T_1) \otimes_{H^n} C(\overline{T_2})$$

<u>tensor product</u>

derived tensor product

$$\begin{array}{ll} C(T_1) \otimes H^n \otimes H^n \otimes C(T_2) & \\ \downarrow & \\ C(T_1) \otimes H^n \otimes C(T_2) & x \otimes a \otimes y \\ \downarrow & \downarrow \\ C(T_1) \otimes C(\overline{T_2}) & xa \otimes y + x \otimes ay \end{array}$$

keep add'ing to kill kernel of maps

derived tensor product $\xrightarrow[\ ]{\overset{9.1.}{\cong}}$ $C(T_1 \cup \overline{T_2})$

Key principle: complete resolutions of a tangles are matchings

$\Rightarrow$ suffices to prove pairing for $T_1 = m_\alpha$, $T_2 = m_\beta$

filtration arg't

$$C(m_\alpha \cup \overline{m_\beta}) \cong C(m_\alpha) \otimes_{H^n} C(\overline{m_\beta})$$

$$C(m_\alpha) = 1_\alpha \cdot H^n \qquad C(\overline{m_\beta}) = H^n \cdot 1_\beta$$

$$\begin{aligned} C(m_\alpha) \otimes_{H^n} C(\overline{m_\beta}) &= 1_\alpha (H^n \otimes_{H^n} H^n) 1_\beta \\ &= 1_\alpha H^n 1_\beta \\ &= C(m_\alpha \cup \overline{m_\beta}) \end{aligned}$$

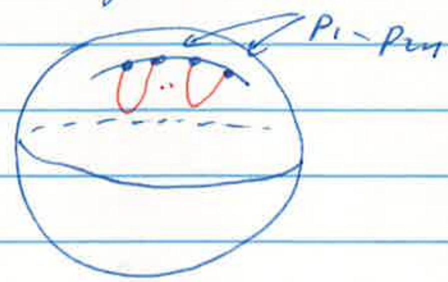
Given $L \subset S^3$, let $\Sigma(S^3, L)$ be double branched cover of L

$$rk(Kh(L)) \geq rk(\widehat{HF}(\Sigma(S^3, L)))$$

Ozsváth - Szabó : $\exists$ spectral sequence

w/ $E_2 = Kh(L)$ $E_\infty = \widehat{HF}(\Sigma(S^3, L))$

view tangles as in B^3



then $\Sigma(B^3, T)$ a 3-manifold w/ ∂

$= \Sigma(S^2, p)$

a surface of genus

$n-1$: S_{n-1}

crossingless matchings $\Longleftrightarrow$ handlebodies

$m_\alpha \longleftrightarrow H_\alpha$

$H_\alpha \cup \overline{H}_\beta = \Sigma(S^3, m_\alpha \cup \overline{m}_\beta)$

$= \# S^1 \times S^2$'s

bordered 3-manifold w/ boundary S is

Y^3 and $\phi : \partial Y \xrightarrow{\cong} S$

or. preserving

idea: algebra $A(S_g) = \bigoplus_{\alpha, \beta} CF(H_\alpha \cup \overline{H}_\beta)$

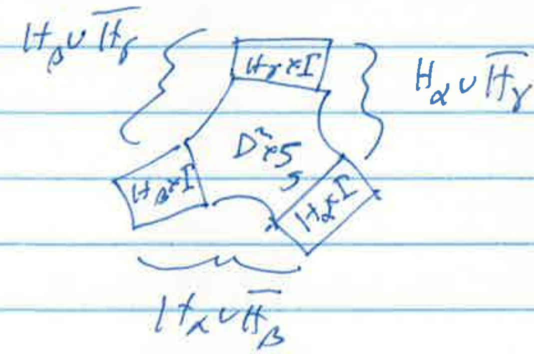
module $C(Y) = \bigoplus_{\alpha} CF(Y \cup \overline{H}_\alpha)$

$C(\overline{Y}) = \bigoplus_{\alpha} CF(H_\alpha \cup \overline{Y})$

pairing: $C(Y_1 \cup \overline{Y_2}) \cong C(Y_1) \otimes_{A(S_g)} C(Y_2)$

this does not work as stated

Problems: 1) need $\mu_2 : CF(H_\alpha \cup \overline{H}_\beta) \otimes CF(H_\beta \cup \overline{H}_\gamma) \to CF(H_\alpha \cup \overline{H}_\gamma)$



μ_2 not associative

but associative upto homotopy

get $\mu_i : A(S_g)^{\otimes i} \to A(S_g)$ $i \geq 2$

satisfying relations

$\rightsquigarrow A_\infty$ algebra

get $m_i : C(Y) \otimes A(S_g)^{\otimes i-1} \to C(Y)$

A_∞ module

$\tilde{\otimes}$ A_∞ tensor product.

Key principle violated here

need to add more handlebodies
to get principle to work

then all this works