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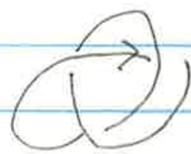
The Jones Polynomial as Euler Characteristic

(joint with Sucharit Sarkar)

A **knot** is a smooth injective

$$K : S^1 \hookrightarrow \mathbb{R}^3 \ (\text{or } S^3)$$

interested up to isotopy



3-crossings

trefoil or 3_1

unknot

figure 8 or 4_1

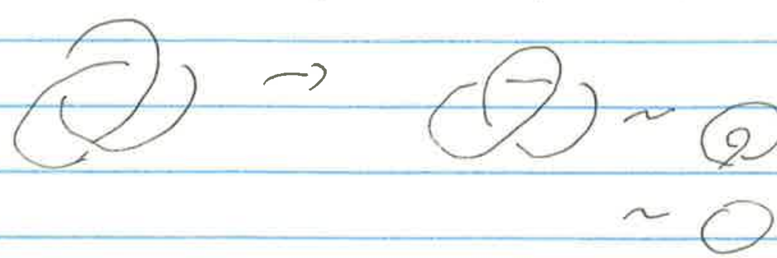
four crossings

Questions:

1. $K_1 \sim K_2$? (isotopic)

   $K_1 \sim U$?

2. $u(K) =$ "unknotting #"
   $=$ min # crossings to change to get U



so $u(3_1) \leq 1$

3. is K alternating ?

   alternating: alternates crossings from over to under

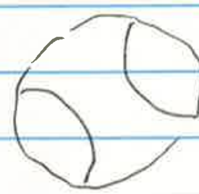
K is alternating if $\exists$ an alternating diagram.

3 invariants:
1) Jones polynomial
2) Khovanov homology
3) Khovanov homotopy type

Jone's Polynomial: (Vaughn Jones 1984)
(this version due to Kauffman ~1986)

$$V : \{\text{knot diag}\} \longrightarrow \mathbb{Z}[q, q^{-1}]$$

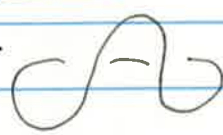
D



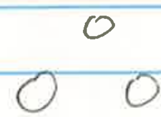
D_0

D_1

example

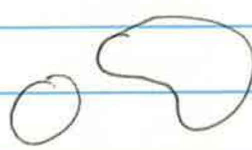


D

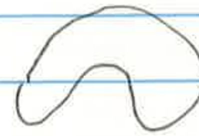


D_{01}

D_{11}



D_{00}



D_{10}

so a diagram with n crossings has

2^n complete resolutions $\leftrightarrow \{0,1\}^n$

states

$$\langle D \rangle = \sum_{\text{states}} (-1)^{|s|} q^{|s|} (q+q^{-1})^{k(s)}$$

$|s| = \sum s$ = # of 1's in S
$k(s)$ = # of circles in D_S

$$V_D(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$$

example above

$$\langle D \rangle = (-1)^{0+0} q^{0+0} (q+q^{-1})^2$$
$$+ (-1)^{1+0} q^{1+0} (q+q^{-1})^1$$
$$+ (-1)^{0+1} q^{0+1} (q+q^{-1})^3$$
$$+ (-1)^{1+1} q^{1+1} (q+q^{-1})^2$$
$$= -q^2 - 1$$

$$V_D(q) = q + q^{-1}$$

<u>Th^{m} (Jones, Kauffman)</u> if D_1, D_2 represent isotopic knots then $V_{D_1}(q) = V_{D_2}(q)$

<u>Examples:</u> $V_U = q + q^{-1}$

$$V_{3_1} = q + q^3 + q^5 - q^9$$

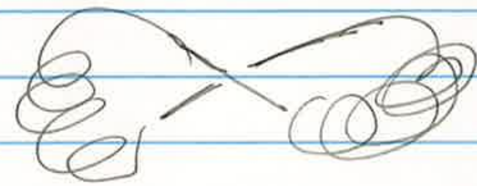
$$V_{4_1} = q^5 - q^{-5}$$

Cor: there are different knots

Cor: $u(3_1) = 1$

Conjecture: if $V_K(q) = V_U(q)$ then $K \sim U$.

A knot diagram is *reduced* if no



Thm: if D is a reduced alternating diagram then D is *minimal*
(no smaller # of crossings)

this was conjecture Tait 1898
proved Kauffman, Murasugi, Thistlethwaite 1987

Thm (Khovanov 1999)
Given a knot K there are (natural) vector spaces $Kh^{i,j}(K)$ st.

$$V_K(q) = \sum (-1)^i q^j \dim(Kh^{i,j}(K))$$

Recall: $u(K)$ = unknot #

$T_{p,q}$ = (p,q) torus knot

$\underline{\text{Th}^{m}}$: $u(T_{p,q}) = \frac{(p-1)(q-1)}{2}$

Conj: Milnor 1968 ($\leq$ easy)

proved: Kronheimer - Mrowka 1993
using gauge theory
(113 pages!)

reproved: Rasmussen 2006 in 28 pages
and combinatorial!
($\exists$ another combinatorial proof by Sarkar)

$\underline{\text{Th}^{m}}$ (Kronheimer - Mrowka, 2010)

if $Kh(K) \cong Kh(U) \Rightarrow K \sim U$

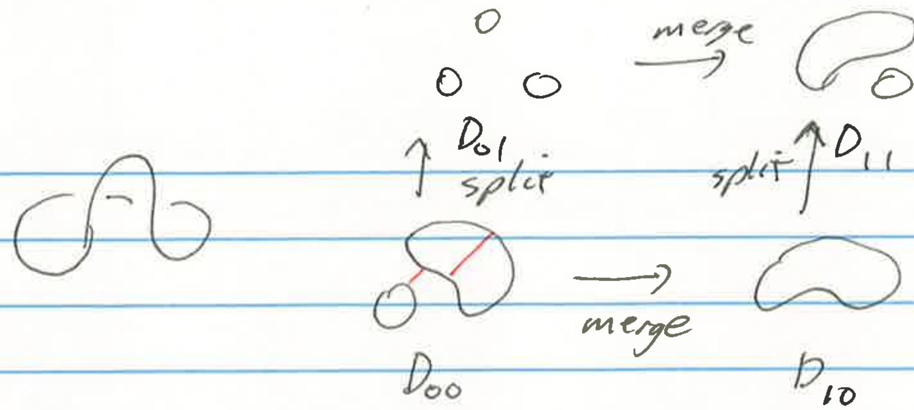
(earlier th^{m} by Grigsby - Wehrli sore using cables...)

$Kh^{*,*}$ is homology of a chain cx $KC^{**}(K)$

$\partial: KC^{i,j} \to KC^{i+1,j}$

Frobenius algebra

$$\begin{cases} V = \mathbb{F}_2[x]/(x^2) & gr(1) = 0 \\ & gr(x) = 2 \\ \Delta: V \to V \otimes V: & x \mapsto x \otimes x \\ & 1 \mapsto 1 \otimes x + x \otimes 1 \end{cases}$$



each cpt of D_{1}, $\longmapsto V$

$\amalg \longmapsto \otimes$

merge $\longmapsto m$

split $\longmapsto \Delta$

$$\begin{tikzcd}
V\otimes V\otimes V \arrow[r, "m\otimes id"] & V\otimes V \\
V\otimes V \arrow[u, "Id\otimes\Delta"] \arrow[r, "m"'] & V \arrow[u, "\Delta"']
\end{tikzcd}$$

direct sum on diagonals

+ need some grading shifts

$V\otimes V \to (V\otimes V\otimes V)\oplus V \to V\otimes V$

chain complex!
homology = Khovanov homology.

Th$\underline{\text{m}}$ (L - Sarkar 2011)

For any $N \gg 0$ $\exists$ CW complex $X^j(K)$ such that $H^{i+N}(X^j(K)) = Kh^{i,j}(K)$

and $X^j(K)$ is well-defined upto stable homotopy equivalence

<u>Cor:</u> $\sum q^i \chi(X^j(K)) = V_K(q)$

1. not Moore spaces (i.e. obvious thing)
2. (Seed) $\exists$ K, K' such that

$$Kh(K) \cong Kh(K')$$
$$\text{but } X(K) \not\simeq X(K')$$

e.g. $11_n 70$, $13_n 2566$

Can use $X^j(K)$ to get a refinement of Rasmussen's invariant $s(K) \in \mathbb{Z}$

Recall: $2u(K) \geq |s(K)|$

(in fact $2g_4(K) \geq |s(K)|$)

$$s(K_1 \# K_2) = s(K_1) + s(K_2)$$

$$Sq^2 : \underset{\underset{Kh^{ij}}{\parallel}}{H^i(X^j(K); \mathbb{Z}/2\mathbb{Z})} \to \underset{\underset{Kh^{i,j+2}}{\parallel}}{H^{i+2}(X^j(K); \mathbb{Z}/2\mathbb{Z})}$$

$\rightsquigarrow$ elt $s^{Sq^2}(K) \in \mathbb{Z}$

$$|s^{Sq^2}(K) - s(K)| \leq 2$$

and $u(K) \geq g_4(K) \geq |s^{Sq^2}(K)|$

Application: $s^{S_q^2}(T_{p,q} \# 9_{42})$

$$\| $$

$$(p-1)(q-1)+2$$

so $u(T_{pq} \# 9_{42}) = \frac{(p-1)(q-1)}{2} + 1$

but $s(9_{42}) = 0$ $\quad \sigma(9_{42}) = 2$

$u(9_{42}) = 1$

$s(T_{p,q}) = \frac{(p-1)(q-1)}{2}$ $\quad \sigma(T_{pq}) = \dots$

$u(T_{3,5}) = 5$