Links in Thickened Surfaces and Virtual Links

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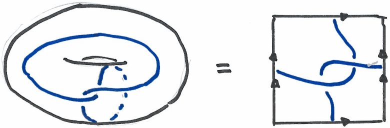
Links in Thickened Surfaces (LTS)

A LTS is a link ℓ in a thickened closed, orientable surface $S \times I$. We usually assume ℓ is oriented.

Links in $\mathbb{S}^2 \times I$ can be identified with links in $\mathbb{S}^3$.

$\ell, \ell' \subset S \times I$ are equivalent if there is an orientation-preserving homeomorphism

$$h : (S \times I, S \times \{0\}) \to (S \times I, S \times \{0\}), \quad h(\ell) = \ell'.$$

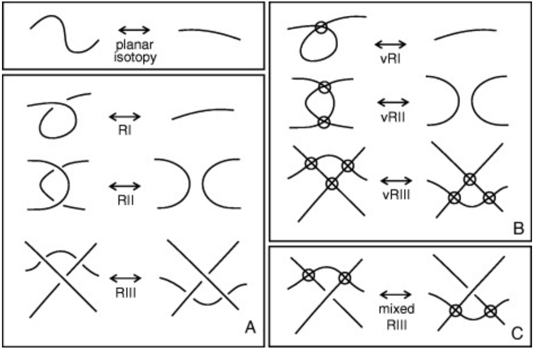


We can regard LTS as diagrams on S modulo Reidemeister moves and orientation-preserving homeomorphisms of S.

Problem: Find invariants of LTS.

Virtual links

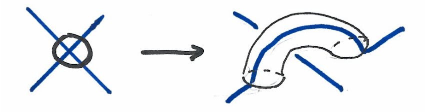
A virtual link is an equivalence class of diagrams in the 2-sphere with classical and virtual crossings, modulo extended Reidemeister moves.



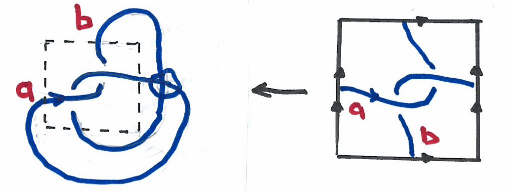
Definition due to Kauffman; extends classical knot theory.

Theorem. (Carter, S. Kamada, Saito) Virtual links correspond to LTS modulo stabilization, that is, the addition and subtraction of hollow handles.

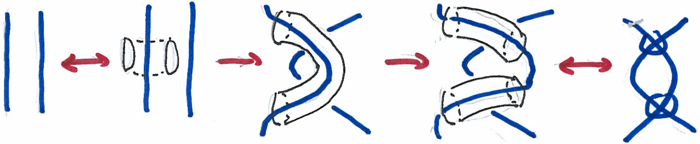
A virtual link diagram becomes a LTS by adding handles at virtual crossings.



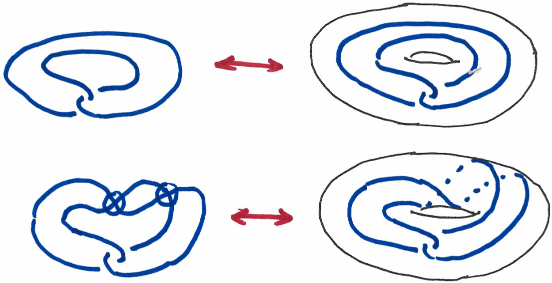
A surface diagram is easily converted to a virtual link, in this example the virtual trefoil.



Realizing a virtual Reidemeister II-move by stabilization of S:



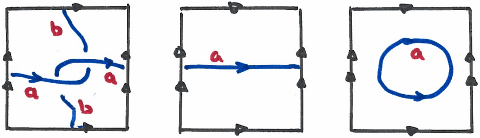
A Dehn twist on S introduces virtual crossings that may be removed by virtual Reidemeister moves.



The virtual link group, which we denote by $\bar{\pi}_\ell$, is obtained from a diagram of ℓ by taking arcs as generators, and the usual Wirtinger relations at classical crossings only.

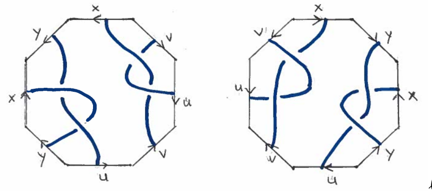
This is an invariant of oriented virtual links, hence an invariant of LTS. Unlike the classical knot group, it is far from complete.

Example: Inequivalent knots in $\mathbb{T}^2 \times I$ corresponding to the trivial virtual knot.



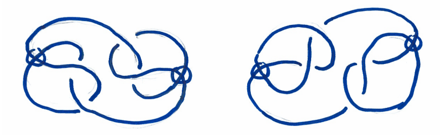
$$\bar{\pi}_{\ell_1} = \langle a, b \mid aa = ba, ab = aa \rangle \cong \mathbb{Z} \cong \bar{\pi}_{\ell_2} \cong \bar{\pi}_{\ell_3}.$$

Example: Flipping the diagram (taking a mirror image and changing crossings) gives an orientation-preserving homeom. of $S \times I$ that takes $S \times \{0\}$ to $S \times \{1\}$.

ℓ  ℓ'

$\bar{\pi}_{\ell'}$ is trivial, but $\bar{\pi}_{\ell}$ is the trefoil knot group.

It follows that ℓ and ℓ' are inequivalent as LTS and as virtual knots, and also that both are nontrivial.



Virtual Genus

Virtual links also correspond bijectively to abstract link diagrams defined by N. Kamada, who introduced the following notion:

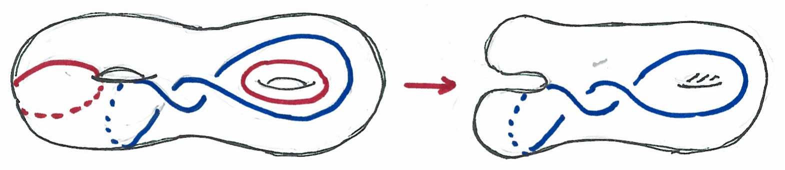
The virtual genus of a virtual knot ℓ is the minimal genus of a surface S supporting a diagram of ℓ. The virtual genus is 0 iff ℓ is a classical link.

Theorem. (G. Kuperberg) A minimal-genus diagram is unique up to Reidemeister moves and orientation-preserving homeomorphism.

Hence any invariant of LTS becomes a virtual link invariant if we apply it to a minimal-genus diagram for the link.

From the proof of Kuperberg's theorem:

If genus(S) $=$v. genus(ℓ) $+ n$ then (after Reidemeister moves) there exists an essential n-component 1-manifold on which we can perform surgery to reduce the genus of S by n.



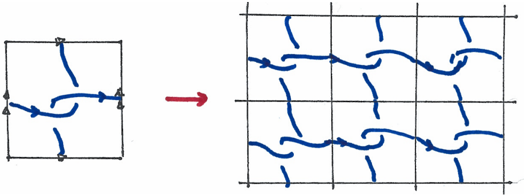
Problem: Find obstructions to genus reduction.

Dye & Kauffman use skein theory and the bracket polynomial.

We use operator groups and Alexander invariants. Our approach is inspired by dynamical systems.

The covering group of a LTS

Lift $\ell \subset S \times I$ to $\tilde{\ell}$ in the universal cover $\tilde{S} \times I \cong \mathbb{R}^2 \times I$.

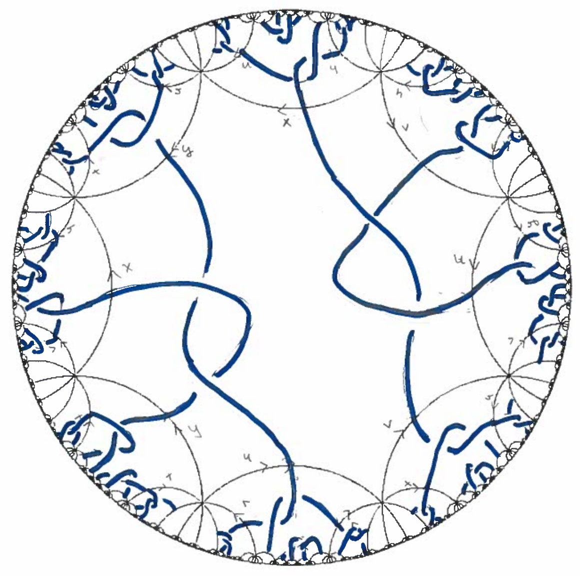


Defintion. The covering group of the LTS ℓ is

$$\tilde{\pi}_\ell = \pi_1((\tilde{S} \times I) \setminus \tilde{\ell}).$$

This is an invariant of LTS, since orientation-preserving homeomorphisms of S lift to $\tilde{S}$.

Example. Lift $\tilde{\ell}$ for our previous genus-2 example.



Operator group structure

$\tilde{\pi}_\ell$ is generally not finitely presented as a group, but it has a finite presentation of another kind:

The group $\Gamma = \pi_1 S = \langle x_1, y_1, \ldots, x_g, y_g \mid \prod [x_i, y_i] \rangle$ of covering transformations of $\tilde{S}$ leaves $\tilde{\ell}$ invariant.

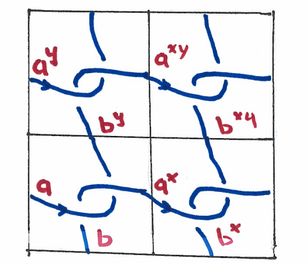
Γ acts on $\tilde{\pi}_\ell$ by automorphisms $g \mapsto g^\gamma$.

$(\tilde{\pi}_\ell, \Gamma)$ is an operator group (Krull, Noether).

It has additional structure: $(u^\gamma)^\eta = u^{\gamma\eta}$, for all $u \in \tilde{\pi}_\ell$ and $\gamma, \eta \in \Gamma$.

The Wirtinger algorithm gives a presentation of $\tilde{\pi}_\ell$ with finitely many Γ-orbits of generators $\{a^\gamma\}$, $\{b^\gamma\}, \ldots$ and relators $\{r^\gamma\}$, $\{s^\gamma\}, \ldots$.

Example. Chain link fence: $\Gamma = \mathbb{Z}^2$



As a group, $\tilde{\pi}_\ell$ has presentation

$$\tilde{\pi}_\ell = \langle a^\gamma, b^\gamma \mid (aa^x)^\gamma = (ba)^\gamma, (a^x b^y)^\gamma = (aa^x)^\gamma,\ (\gamma \in \Gamma)\ \rangle$$

As a Γ-operator group, it has a finite presentation

$$\tilde{\pi}_\ell = \langle a, b \mid aa^x = ba, a^x b^y = aa^x \rangle_\Gamma \cong \langle a \mid a^x a^y a^{xy} = aa^x a^y \rangle_\Gamma$$

Some properties

We say $\ell \subset S \times I$ is trivial if its components bound disjoint discs.

Theorem. A LTS $\ell \subset S \times I$ is trivial iff $\tilde{\pi}_\ell \cong \langle a_1, \ldots, a_d \mid \rangle_\Gamma$.

• Trivializing the action of Γ produces the virtual knot group $\bar{\pi}_\ell$.
In the above example:

$$\tilde{\pi}_\ell = \langle a, b \mid aa^x = ba, a^x b^y = aa^x \rangle_\Gamma$$

$$\bar{\pi}_\ell = \langle a, b \mid aa = ba, ab = aa \rangle$$

• For genus $g = 0$ ($S = \mathbb{S}^2$), we have $\tilde{S} = S$ and $\Gamma = \langle 1 \rangle$. Then $\tilde{\pi}_\ell \cong \pi_\ell := \pi_1(\mathbb{S}^3 \setminus \ell)$, with trivial operator action.

• For genus $g > 0$ we recover $\pi_\ell := \pi_1(S \times I \setminus \ell)$ from any Γ-operator group presentation of $\tilde{\pi}_\ell$ by

(1) adding standard generators $x_1, y_1, \ldots, x_g, y_g$ of Γ and the relator $\prod [x_i, y_i]$; and

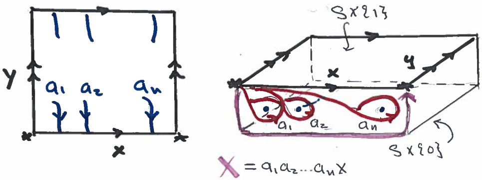
(2) taking u^γ to denote $\gamma u \gamma^{-1}$.

For our example,
$\pi_\ell = \langle a, b, x, y \mid aa^x = ba, a^x b^y = aa^x, [x, y] = 0 \rangle$

By Waldhausen, $S \times I \setminus \ell$ is determined by the peripheral group system, comprising π_ℓ and π_1(boundary components).

In the above presentation of π_ℓ, $\pi_1(S \times \{1\})$ is the subgroup generated by $x_1, y_1, \ldots, x_g, y_g$. Γ-group isomorphisms of $\tilde{\pi}_\ell$ preserve this peripheral information.

$\pi_1(S \times \{0\})$ is generated by elements $X_1, Y_1, \ldots, X_g, Y_g$. Here $X_i = A_i x_i$ where A_i is the product of generators corresponding to arcs meeting the corresponding edge of the fundamental domain, and similarly for Y_i.



The fundamental group of the boundary of a link component neighborhood is generated by a meridian and longitude, also easily read from the diagram.

Symplectic rank and virtual genus

$H_1 S \cong \mathbb{Z}^{2g}$ has a symplectic form corresponding to the intersection pairing of generators $x_1, y_1, \ldots, x_g, y_g$.

The symplectic rank $\mathrm{rk}_s(V)$ of a submodule V of $H_1 S$ is the dimension of a max. symplectic subspace of $V \otimes \mathbb{R} \subset \mathbb{R}^{2g}$.

For a Γ-group presentation P of $\tilde{\pi}_\ell$, let V_P be the submodule of $H_1 S$ generated by operators that appear in the relations.

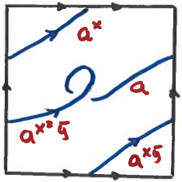
Definition. The symplectic rank of $\tilde{\pi}_\ell$ is the minimum $\mathrm{rk}_s(V_P)$ over all presentations.

Theorem. Let ℓ be a non-split virtual link. For any representative $\ell \subset S \times I$,

$$\tfrac{1}{2} \text{ symplectic rank of } \tilde{\pi}_\ell = \text{virtual genus } (\ell).$$

Thus $\frac{1}{2}\mathrm{rk}_s(V_P) \geq$ virtual genus (ℓ).

Example. Let $k \subset \mathbb{T} \times I$ be given by



$\tilde{\pi}_k = \langle a \mid a^{x^2\bar{y}} = a \rangle_\Gamma$ is nontrivial, so k is nontrivial in $S \times I$.

$V_P = \langle x^2\bar{y} \rangle$ has symplectic rank 0, so the virtual genus is 0.
In fact, k is trivial as a virtual knot.

Proof of Theorem is based on Kuperburg's theorem and work of Waldhausen.

Alexander invariants from $\tilde{\pi}_\ell$

Let $\ell = \ell_1 \cup \ldots \cup \ell_d \subset S \times I$.

Let $\epsilon : \tilde{\pi}_\ell \to \mathbb{Z}^d \cong \langle t_1, \ldots, t_d \mid [t_i, t_j] = 1 \ \forall\, i, j \rangle$ be the homomorphism sending the meridians of lifts of ℓ_i to t_i.

$K = \ker \epsilon$

$M = K/K'$ is a module over $\mathbb{Z}[\Gamma \times \mathbb{Z}^d]$, a natural choice of Alexander module for LTS.

For polynomial invariants, we want a *Noetherian* module.

Let $\bar{M} = M / \{\gamma a - \eta a : \gamma\eta^{-1} \in \Gamma', a \in K/K'\}$.

$\bar{M}$ is a module over $\mathbb{Z}[H_1\Gamma \times \mathbb{Z}^d] \cong R[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$, where $R = \mathbb{Z}[\mathbb{Z}^{2g}]$.

Lemma. $\bar{M}$ is presented by a square matrix A with entries in $R[t^{\pm 1}]$.

We define the Alexander polynomial of a LTS to be

$$\Delta_\ell(t_1, \ldots, t_d) = \det A.$$

Coefficients are in $\mathbb{Z}[\mathbb{Z}^{2g}]$.

An orientation-preserving homeomorphism ϕ of S induces a symplectic automorphism $\phi_\#$ of $H_1 S$.

$\Delta_\ell(t_1, \ldots, t_d)$ is well defined up to a symplectic change of basis

$$\sum c_{i,n}\, t_i^n \to \sum \phi_\#(c_{i,n})\, t_i^n$$

and multiplication by units in $R[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.

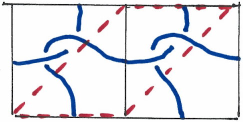
Example. For the chain link fence,

$$\tilde{\pi}_\ell \cong \langle a \mid a^x a^y a^{xy} = a a^x a^y \rangle_\Gamma.$$

We have $\epsilon : a \mapsto t$. Applying Fox calculus:

$M \cong \langle a \mid (x + ty + t^2 xy)a = (1 + tx + t^2 y)a \rangle$

$\Delta_\ell(t) = (xy - y)t^2 + (y - x)t + (x - 1)$

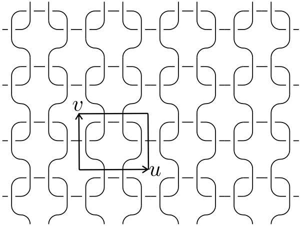


A Dehn twist induces $\phi_\# : x \mapsto xy,\ y \mapsto y$.

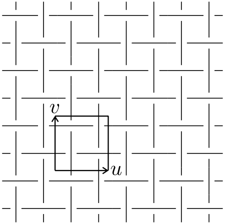
$\Delta_\ell(t)$ is equivalent to $(xy^2 - y)t^2 + (y - xy)t + (xy - 1)$.

Textiles

For $g = 1$, $\tilde{\ell}$ is a doubly periodic textile structure (Grishanov, Meshkov & Omelchenko; Grishanov, Meshkov & Vassiliev; Morton & Grishanov).



Single jersey.

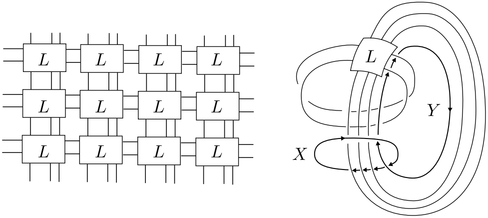


Plain weave.

(Diagrams from Morton & Grishanov.)

These authors study textiles via the quotient link in $\mathbb{T}^2 \times I$.

Morton and Grishanov obtain a link $\mathcal{L}$ in $\mathbb{S}^3$ from the periodic textile by projecting to a link in the thickened torus and adding components X, Y in the complement.



Our invariant Δ_ℓ is the multivariable Alexander polynomial of $\mathcal{L}$ considered by Morton and Grishanov.

Our method of computation is much easier, basically because our x and y both lie on the same surface boundary.

Symplectic rank and virtual genus, II

Definition. The symplectic rank of Δ_ℓ is $\mathrm{rk}_s(W)$, where W is the subspace of $\mathbb{R}^{2g}$ spanned by ratios of coefficients of Δ_ℓ.

Prop. This is invariant under symplectic change of basis.

Theorem. virtual genus $(\ell) \geq \frac{1}{2}$ symplectic rank of Δ_ℓ.

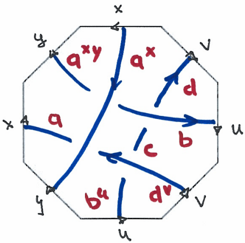
Example. For the chain link fence we found

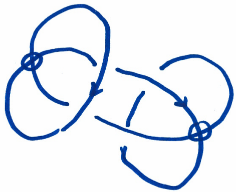
$$\Delta_\ell(t) = (xy - y)t^2 + (y - x)t + (x - 1).$$

$W = \mathrm{span}\{x, y\}$, $\mathrm{rk}_s(W) = 2$, so virtual trefoil has genus ≥ 1.

Since we have a genus 1 diagram, the virtual genus is 1 (already well known!)

Example: Kishino's knot





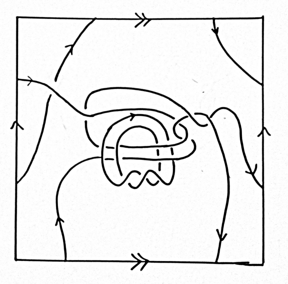
$\tilde{\pi}_k = \langle a, b, c, d \mid a^x b = a^{xy} a^x, a^x d^v = a a^x, bd = cb, d^v b^u = c d^v \rangle$

$\Delta_k = (x - uvx)t^2 + (1 - v - x + uvx - vxy - uvxy)t + (vxy - v)$

$W = \text{span}\{x, y, u, v\} = \mathbb{R}^4$. Δ_k has symplectic rank 4, so virtual genus of k is ≥ 2. Thus virtual genus = 2.

This was shown previously by Dye and Kauffman with other techniques.

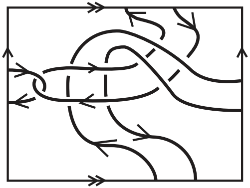
Example. Stoimenow's link



Dye and Kauffman give this example, due to A. Stoimenow, for which their methods do not determine virtual genus.

The symplectic rank of Δ_ℓ is equal to 2. Since the link has a diagram on a torus, we see the link has virtual genus 1.

Example. Virtual trefoil double



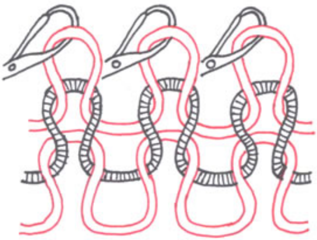
This is a satellite of the virtual trefoil (chain link fence).

Theorem. [Silver-W] The virtual genus of a satellite $\tilde{k}$ of k is equal to the virtual genus of k.

For this example, $\Delta_{\tilde{k}} = (t-1)(xy-1)^2$ has symplectic rank 0. It does not detect virtual genus, which is 1.

Note that the symplectic rank of the Γ-group presentation from this diagram must be 2.

Thank you for listening.



Single jersey (image: Woolmark Co.)