Semi-global Kuranishi charts & the global def$^{\underline{n}}$ of contact homology

(Joint w/ Erkao Bao)

(M^{2n+1}, ξ) = closed contact manifold
$\xi = \ker \alpha \qquad \alpha \wedge (d\alpha)^n > 0$

Goal: Define contact homology, originally proposed by Eliashberg-Givental-Hofer and prove its an invariant

$\exists$ other approaches: Hofer-Wysoki-Zehnder
Pardon

1. Intro

α = nondegenerate contact form
R_α = Reeb vector field for α
$$R_\alpha \lrcorner\, d\alpha = 0$$
$$\alpha(R_\alpha) = 0$$

$$\mathcal{P}_\alpha^{good} = \{\text{"good" Reeb orbits of } R_\alpha\}$$

J = almost complex str on $\mathbb{R} \times M$
adapted to α: $J \frac{\partial}{\partial s} = R_\alpha$
($\mathbb{R}$-invariant) $J\xi = \xi$
$d\alpha(v, Jv) > 0 \quad \forall\, v \neq 0 \in \xi$

J does not need to be generic

algebra: free unital supercommut. algebra A over $\mathbb{Q}$ generated by P_α^{good}

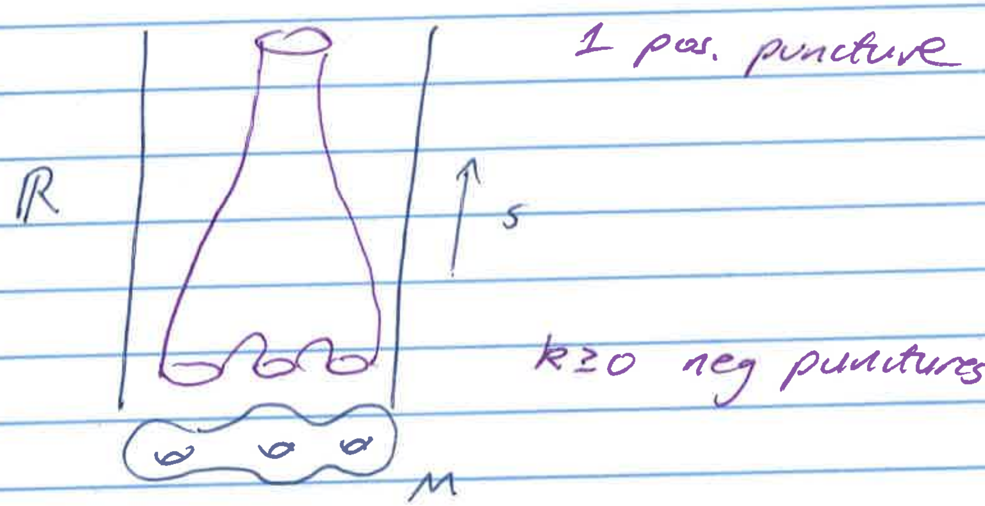
differential: $\approx$ counts J-holomorphic curves

$$u : (\dot{F}, j) \longrightarrow (\mathbb{R}\times M, J)$$

(F, j) closed Riemann surface

$\dot{F} = F - pts$

Ideally:



at $\pm\infty$ asymptotic to cyl $\mathbb{R}\times\gamma$

γ Reeb orbit

for $+\infty$ γ_+

$-\infty$ $\gamma_{-,1}, \ldots \gamma_{-,k}$

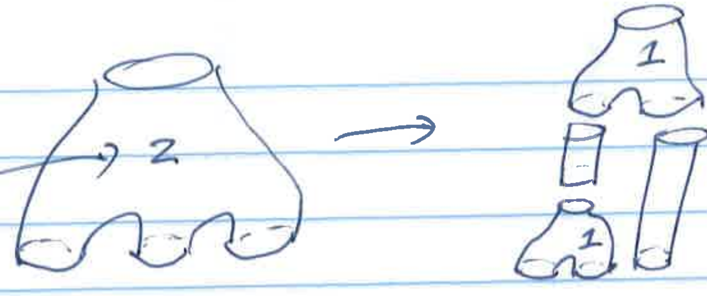
Fredholm index $= 1$

$$\partial\gamma_+ = (\text{count})\, \gamma_{-,1} \cdots \gamma_{-,k}$$

...

$\partial^2 = 0$:

Fredholm index



mod out by $\mathbb{R}$ translation
get 1-mfd

has 2 boundary points
so $\partial^2 = 0$.

In reality:

total index 1

shouldn't exist
but frequently
has to exist
(e.g. $\exists$ some curve
of > index &
some branched cover
of it has neg index)

need to count but how?

2. Semi-global Kuranishi charts

Fix $(F, \underset{\text{pts}}{p})$ & $\dot{F}$ assume $\chi(\dot{F}) < 0$

$\text{Teich}(\dot{F})$ = Teichmüller space of $\dot{F}$
viewed as equivalence
classes of cx. strs on $\dot{F}$

$\mathcal{U} \subset \mathrm{Teich}(\dot{F}) \rightsquigarrow$ slice $\tilde{\mathcal{U}}$ of $\mathcal{U}$, i.e. smooth choice of complex str. for each $[j] \in \mathcal{U}$.

Banach space setup

$$\pi : \mathcal{E} \longrightarrow \mathcal{B}$$

$$\mathcal{B} = \mathcal{B}_{\tilde{\mathcal{U}}}(\dot{F}, \mathbb{R}\times M, \gamma_+, \gamma_-)$$

$$= \{ ((F, j, p), u) \mid j \in \tilde{\mathcal{U}},\ u \in W^{k+1,p}_{\delta}(\dot{F}, \mathbb{R}\times M, \gamma_+, \gamma_-) \}$$

fiber $\mathcal{E}_{(\mathcal{F}, u)} = W^{k,p}_{\delta}(\dot{F}, \Lambda^{0,1} u^* T(\mathbb{R}\times M))$

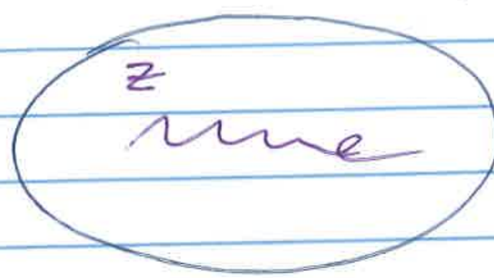
J-holomorphic satisfies $\bar\partial u = 0$

$$\bar\partial u = \frac{1}{2}(du + J(u)\, du \circ j)$$

$\bar\partial$ is a section of bundle

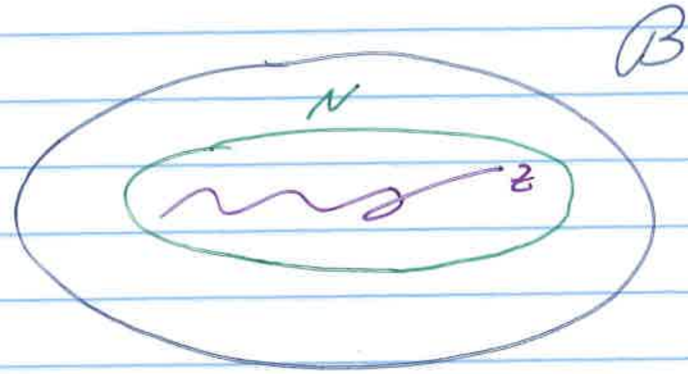
Suppose $Z = \bar\partial^{-1}(0)$ is compact

$\mathcal{B}$



if $\mathcal{Z}$ compact

Th$^{\underline{m}}$: For a suff. small nbhd $N(\mathcal{Z})$ of $\mathcal{Z}$
$\exists$ a finite rank subbundle
$E = E^{\ell, \varepsilon} \to N(\mathcal{Z})$ which is transverse to $\bar\partial$
$\cap$
$\mathcal{E}$



let $V = \bar\partial^{-1}(E)$ $\quad$ V is finite dim'l manifold

$$\begin{tikzcd} E|_V \arrow[r] & V \arrow[l, bend right, "\bar\partial"'] \end{tikzcd}$$
$$V \supset \mathcal{Z}$$

choose generic section s st. $s|_{\partial V} = 0$

$\Rightarrow$ $\pitchfork$ replacement for $\mathcal{Z}$ is $\bar\partial \cap s$

① <u>asymptotic operator</u>

Near an end γ of u, $\bar\partial u = 0$ looks like

$$\left(\frac{\partial}{\partial s} + \underbrace{\left(J\frac{\partial}{\partial t_1} + S\right)}_{-A}\right)(s, t, \eta(s,t)) = 0$$

S: symmetric; s: $\mathbb{R}$; t: param γ

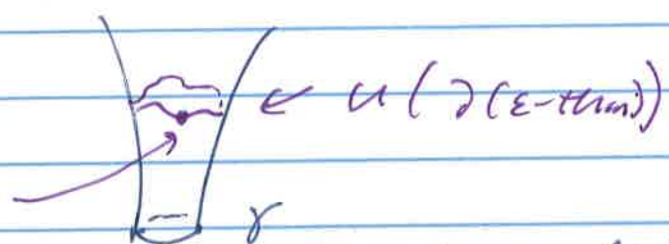
asymptotic operator $\to -A$

Eigenvalues $\dots \le -\lambda_{-2} \le \lambda_{-1} \le 0 < \lambda_1 \le \lambda_2 \le \dots$

Eigen functions $f_{-2}, f_{-1}, f_1, f_2, \dots$

(2) $\chi(\dot F) < 0 \Rightarrow \exists$ hyp. metric g compatible w/ j

(3) On each end limiting to γ



choose $\beta(s)$ bump function

big $\rightarrow \ell, \varepsilon \leftarrow$ small

$$E^{\ell,\varepsilon}_{u,\gamma,\pm} = \mathbb{R}\langle \hat f_1 \dots \hat f_\ell \rangle$$

$E^{\ell,\varepsilon}_u = \oplus E^{\ell,\varepsilon}_{u,\gamma,\pm}$

$$\tilde f_i(s,t) = \beta(s)\, f_i(t)(ds - i\,dt)$$

(**) $E^{\ell,\varepsilon}_{\dots}$ was chosen geometrically using u & only depends on

- hyp geom of $\dot F$
- Image of u.

$\Rightarrow$ does not choice of slice $\tilde{\mathcal{U}}$

(4) Check $\pitchfork$ $\quad \overline{\bar\partial u = 0}$

linearized $\bar\partial$-operator.

if $E_u^{l,\varepsilon} + \operatorname{Im} D_u \subsetneq W_\delta^{k,p}(\dot{F}, \Lambda^{0,1} u^* T(\mathbb{R}\times M))$

then $\exists\; 0 \neq \zeta \in \operatorname{Ker} D_u^*$ which is L^2-orthogonal to $E_u^{l,\varepsilon}$

But then $\zeta(s,t) = \sum_{\lambda_i > 0} c_i e^{\lambda_i s} f_i(t) \otimes (ds - i\,dt)$

some $c_i \neq 0$

if $l \gg 0$ $\Rightarrow\!\Leftarrow$

each point has an l that works & since $\pitchfork$ is open you get a finite # of open sets that cover Z and has an l that works

choose max l over all these

In general, get $(\pi : \mathcal{E} \to \mathcal{V}, \mathfrak{s})$

orbi bundle

$\mathfrak{s}$ = multi section

st. $\mathbb{V} \supset K$

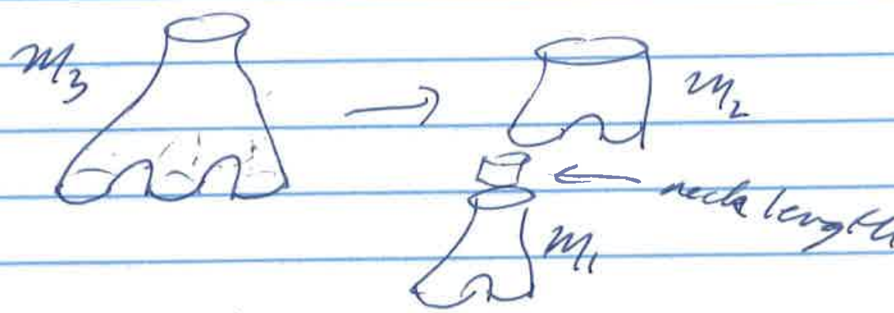
large compact subset of $\mathcal{Z}$ ← call $\mathcal{M}$

3. Semi-global Kuranishi str ∃ ordering of moduli spaces $\mathcal{M}_1, \mathcal{M}_2, \ldots$

"Boundary"

$\partial \mathcal{M}_3 \subset \mathcal{M}_1 \times \mathcal{M}_2$

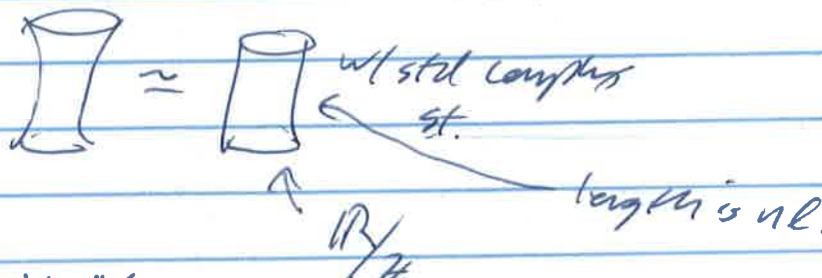
st. each stratum of $\partial \mathcal{M}_i \subset$ product of $\mathcal{M}_j$, $j < i$



all perturbations chosen here

so chosen too.

neck length (nl) = modulus of ε-thin annulus



region to interpolate s_3 to $s_1 \times s_2$

can do this arbitrarily