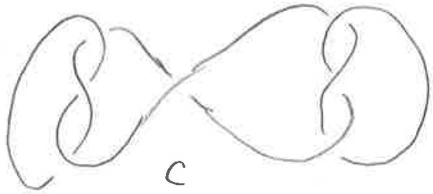
Cosmetic Surgeries in L-spaces and nugatory crossings

joint with Lidman


C

$K \subset S^3$ or

Q: $K^+ \cong K^-$

D = crossing disk
∂D = crossing circle

C is nugatory iff. ∂D bounds an embedded disk in $S^3 \setminus K$

"Nugatory" Cosmetic Crossing Conjecture (X. S. Lin)

If K admits a crossing change at C which preserves oriented isotopy type of K, C is nugatory.

What is known:

- unknot
- 2-bridge knots
- fibered
- Whitehead doubles

Balm Friedl-Kotf, Powell: $g(K) = 1$, K admits cosmetic cr change. $\Rightarrow$ $\Delta_K(t) = f(t) \cdot f(t^{-1})$.

Q: Alternating knots.

1st Tait Conj. (1980s) Reduced alt. diag's are minimal and min. diag. of prime alt. knot alternates.

Thm (Manolescu-Ozsváth; Lee)

Alternating and QA are $\widetilde{Kh}$-thin.

reduced Khovanov homology of $L \subset S^3$

$\widetilde{Kh}^{i,j}(L)$ coeff. in $\mathbb{Z}_2$, L is $\widetilde{Kh}$-thin if supported along single diagonal.

Thm (Ozsváth-Szabó) K is $\widetilde{Kh}$-thin $\Rightarrow \Sigma(K)$ is an L-space.

A $\mathbb{Q}HS^3$ Y is an L-space if $|H_1(Y;\mathbb{Z})| = \operatorname{rk} \widehat{HF}(Y)$

Here $H_1(\Sigma(K)) \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k$ $\quad (*)$

Thm (Lidman-M): Let K be a knot w/ $\Sigma(K)$ an L-space. If each d_i in $(*)$ is squarefree, K satisfies C.C.C.

Applications: ① small knots excluding 2 bridge, fibered. thm $\Rightarrow$ all but ten knots.

② Pretzel knots $P = (-p, q, r)$ $p \geq 4$ even $q, r \geq 3$ odd

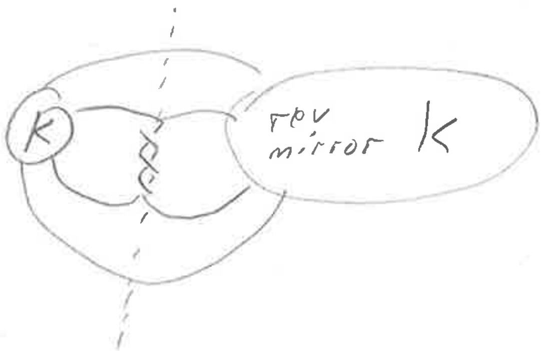
Gabai $\Rightarrow$ P non-fibered, hyperbolic, 3 bridge.

$q = p-1 \Rightarrow P$ is QA

③ Banach sets of L-space surgeries. K strongly invertible L-space knots, p = SF odd number $p/q \geq 2g(K)+1$

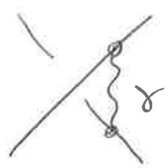
$S^3_{p/q}(K) \simeq \Sigma(I_{p/q})$

(4) Thm: $\exists$ ∞ family of hyperbolic, non-fibered, 3 bridge knots of fixed det satisfying C.C.C.



check $\widetilde{Kh}$-thin, $\widehat{HFK}$:

Stretch Main Thm Assume K admits cosmetic crossing change at C s.t. $K^+ \simeq K^-$



γ = cosmetic arc

$\tilde{\gamma}$ = knot in $\Sigma(K)$

Suppose $[\tilde{\gamma}] = 0$ in $\Sigma(K)$.

Thm: K null-homologous in L-space Y. Then

$$Y_{p/q}(K) \cong Y_{p/q}(U) \implies K \simeq U.$$

$M = \Sigma(K) \setminus \tilde{\gamma}$

$M(\alpha) = \Sigma(K^+)$

$M(\beta) = \Sigma(K^-)$

$\Delta(\alpha, \beta) = 2$, half int. surg. on $\tilde{\gamma} \rightsquigarrow \Sigma(K^-)$

" " $u \rightsquigarrow \Sigma(K^+)$

$\tilde{\gamma} \simeq u$.

<u>Lemma</u>: $\tilde{\gamma}$ is an unknot $\Rightarrow$ C is nugatory.

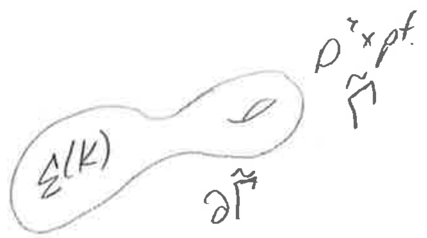
<u>idea</u> follows after application of $\mathbb{Z}_2$-equivariant Dehn's lemma.

$\mathbb{Z}_2 \overset{\tau}{\curvearrowright} M^3$, $C \subset \partial M$, $C \simeq 0$ in M

$\tau(c) \cap c = \emptyset$ or $\tau(c) = c$

$\Rightarrow \exists D$ with $\partial D = C$ s.t.

$$M = \Sigma(K) \setminus \tilde{\gamma} \cong \Sigma(K) \# D^2 \times S^1 \cong \Sigma(B, T)$$



i) $\tau(\tilde{\Gamma}) \cap \tilde{\Gamma}$

ii) $\tau(\tilde{\Gamma}) \cap \tilde{\Gamma}$

<u>Prop</u>: $[\tilde{\gamma}] = 0$. Choose basis (λ_n, μ) for $H_1(\partial M)$

$$H_1(\partial M) \longrightarrow H_1(M) = \mathbb{Z} \oplus H$$

$\alpha = p\mu + q\lambda_m$

$\beta = r\mu + s\lambda_n$

Fact: λ_n admits order H_1 of filling.

$$|H_1 M(\alpha)| = c_n \Delta(\alpha, \lambda_n)$$
$$\quad\ \|\qquad\qquad\qquad \|$$
$$|H_1 M(\beta)| = c_m \Delta(\beta, \lambda_m)$$

$\Rightarrow p = r$

$\Delta(\alpha, \beta) = 2$.

$$2 = p(q-s) \implies p = 1 \text{ or } \cancel{p = 2}$$

$$\det K = |H_1 \Sigma(K)| = \Delta(\alpha, \gamma_m) c_m$$
$$= c_m$$
$$= \operatorname{ord}_H i_*(\gamma_m) \cdot |H|$$

Subcases: if $\det(K)$ is squarefree (each d_i prime)

$\operatorname{ord}_H i_*(\gamma_m) = 1$

$\gamma_m \simeq \tilde{\gamma}$ in filling $\implies |\tilde{\gamma}| = 0$

Conj: ① if $\Sigma(Y, K)$ is an L-space $\implies Y$ is an L-space.

② Only $\mathbb{Z}HS^3$ L-spaces are $S^3 \# P^3$.

Q: What characterizes $\Sigma(S^3, K)$ being an L-space?