Matt Clay
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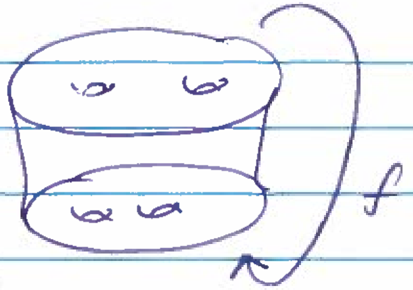
ℓ^2-torsion of free by cyclic groups



$f : \Sigma \to \Sigma$ homeo

$M_f = \Sigma \times [0,1] / (x,1) \sim (f(x),0)$

Thurston: $M_f = \Big(\underset{\substack{\uparrow \\ \text{Seifert fibered}}}{\bigcup S_i}\Big) \cup \Big(\underset{\substack{\uparrow \\ \text{hyperbolic}}}{\bigcup H_{i'}}\Big)$

JSJ: this is unique

Thm (Kojima–McShane '14)

If $f : \Sigma \to \Sigma$ is pseudo-Anosov

$$\mathrm{vol}(M_f) \le 3\pi|\chi(\Sigma)| \log \underset{\substack{\uparrow \\ \text{stretch factor.}}}{\lambda(f)}$$

$$\lambda(f) = \sup_{\gamma} \lim_{k\to\infty} \sqrt[k]{\underset{\substack{\uparrow \\ \sigma \text{ hyperbolic on } \Sigma}}{\ell_\sigma}[f^k(\gamma)]}$$

(don't need sup here but OK)

$$1 \to \mathbb{F} \to G \to \mathbb{Z} \to 1$$

$\mathbb{F}$ ↑ f.g. non-abelian free group

G is a free by cyclic group

<u>note</u>: $G \cong \mathbb{F} \rtimes_{\Phi} \mathbb{Z} = \langle \mathbb{F}, t \mid t^{-1} x t = \Phi(x),\ x \in \mathbb{F} \rangle$

$\Phi \in \mathrm{Aut}(\mathbb{F})$ only depends on $\phi = [\Phi] \in \mathrm{Out}(\mathbb{F})$

denote group by G_ϕ

ℓ^2-torsion : behaves like χ
(Lück) (more later)

set up $G = \mathbb{F}$ or $\pi_1(\Sigma)$ (any residually finite gp)

$$\Phi \in \mathrm{Aut}(G)$$
$$\downarrow$$

ℓ^2-torsion $\rho^{(2)}(G \rtimes_\Phi \mathbb{Z}) \in \mathbb{R}$

__Fact__: only depends on $G \rtimes_\Phi \mathbb{Z}$, not Φ.

__Th$^{\underline{m}}$ (Lück-Schick '99)__:

$$M_f = (\cup S_i) \cup (\cup H_i)$$

$$-\rho^{(2)}(\pi_1(M_f)) = \frac{1}{6\pi} \sum \mathrm{vol}\, H_i$$

__Main Th$^{\underline{m}}$__: Given upper bound on $-\rho^{(2)}(G_\phi)$ in terms of dynamics of ϕ

note: $-\rho^{(2)}(G_\phi) \geq 0$ (Lück)

Relative Train tracks (Bestvina-Handel '92)

- $f : \Gamma \to \Gamma$ homotopy equiv. inducing ϕ on $\pi_1(\Gamma) \cong \mathbb{F}$

- $\phi = \Gamma_0 \subseteq \Gamma_1 \subseteq \dots \subseteq \Gamma_S = \Gamma$ where

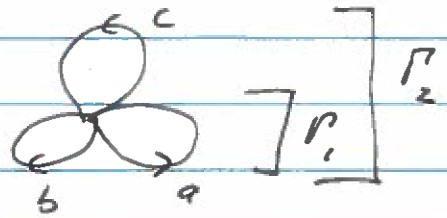
  $f(\Gamma_s) \subseteq \Gamma_s$

  \+ 3 properties

example: $f: a \mapsto ab$

$b \mapsto ab^2$

$c \mapsto caba^{-1}b^{-1}$



$[M(f)]_{i,j}$ = # times $e_j^{\pm}$ appears in $f(e_i)$

$$M(f) = \begin{bmatrix} \overset{M(f)_1}{\boxed{\begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix}}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 2 & 2 \end{matrix} & \underset{M(f)_2}{\boxed{1}} \end{bmatrix}$$

$M(f)_s$ = submatrix corresponding to $\Gamma_s - \Gamma_{s-1}$

$M(f)$ is lower block triangular w/ $M(f)_s$ on diagonal

$$\text{RTT} \Rightarrow \underset{\uparrow}{M(f^k)_s} = M(f)_s^k$$

iterate and tighten.

$M(f)_s$ is either zero or irreducible

$\rightsquigarrow$

$\lambda(S)_s$

Perron-Frobenius eigenvalue

$\mathcal{EG}(f) = \{ s \mid \lambda(f)_s > 1 \}$

Thm: $$-\rho^{(2)}(G_\phi) \le \sum_{s \in \mathcal{EG}(f)} n_s \log \lambda(f)_s$$

n edges in $\Gamma_s - \Gamma_{s-1}$ size of $M(f)_s$

Cor: If $f : \Gamma \to \Gamma$ is irreducible

$$-\rho^{(2)}(G_\phi) \le 3|\chi(\Gamma)| \log \lambda(f)$$

$$\sup_x \lim_{k \to \infty} \sqrt[k]{\| \phi^k(x) \|}$$

cyclic length w.r.t. basis

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torsion: determinant of an acyclic chain complex of finite dim'l vector space

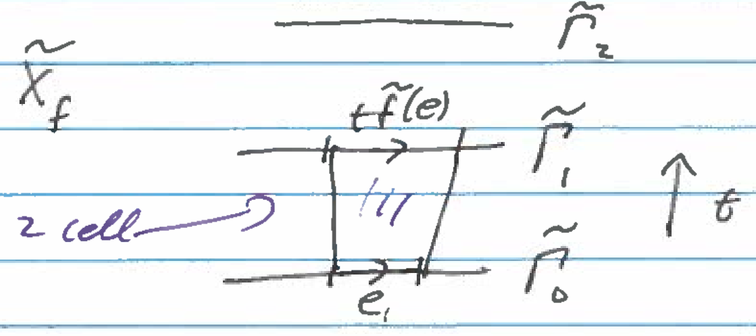
$$\begin{tikzcd}
{} \arrow[ddr, "\cong"] & A_{n+1} \arrow[ddr, "\cong"] & A_n \arrow[ddr, "\cong"] & A_{n-1} \arrow[r, "\cong"] & {} \\
 & \oplus & \oplus & \oplus & \\
 & B_{n+1} \rightarrow 0 & B_n \rightarrow 0 & B_{n-1} \rightarrow 0 &
\end{tikzcd}$$

$$C_* \quad \cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$$

$$\rho(C_*) = \sum (-1)^{n-1} \log |\det (A_n \xrightarrow{\cong} B_{n-1})|$$

Apply to chain complex for universal cover of $X_f = \Gamma \times [0,1] / (x,1) \sim (f(x),0)$

fix a lift $\tilde f : \tilde\Gamma \to \tilde\Gamma$ and a corresponding $t \in G_\phi$



$A = \mathbb{C}[G_\phi]$ $\quad$ E = # edges in Γ

V = # vertices in Γ

$$0 \to \underset{\substack{\uparrow \\ \#\text{ 2-cells}}}{A^E} \to \underset{\substack{\uparrow \\ \text{horizontal} \\ \text{edges}}}{A^E} \oplus \underset{\substack{\uparrow \\ \text{vertical} \\ \text{edges}}}{A^V} \longrightarrow \underset{\substack{\uparrow \\ \#\text{ vertices}}}{A^V} \to 0$$

boundary of a 2-cell over e is

$$\underbrace{e - t\tilde f(e)}_{\substack{\text{horizontal} \\ \partial_h}} \oplus \underbrace{\partial(e)}_{\substack{\text{vertical} \\ \partial_v}}$$

$$\rho(G_\phi) = -\log \det(\partial_h)$$

$\hookleftarrow A^E$ so don't to need to pass to completion

$\bar A = \ell^2(G_\phi)$

$\partial_h : [\ell^{(2)}(G_\phi)]^E \to [\ell^2(G_\phi)]^E$

now apply Fuglede-Kadison determinant

∂_n is right multiplication by

$I - tJ(f) \in \mathrm{Mat}_E(\mathbb{Z}[G_\phi])$

$[J(f)]_{i,j}$ = coeff of e_j in $\tilde{f}(e_i) \in \mathbb{Z}[\mathbb{F}]$

↑ Fox calculus

$(J(f))_{\ell^1} = M(f)$

↑ replace by ℓ^1 norm

lower block diagonal w/ $J(f)_s$ on diagonal

Thm $-\rho^{(2)}(G_\phi) = \log(\det(I - tJ(f)))$

$= \sum \log \det(I - tJ(f)_s)$