Geometrically similar knots and 3–manifolds

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Geometric inverse problems

Question (Weyl, Bers, Kac): Can you hear the shape of a drum?

For this talk,

- *Drum* means a 3–manifold, often the complement of a knot in S^3.
- *Shape* means the complete hyperbolic metric.
- *Hear* means to determine from the spectrum of lengths of closed geodesics, listed with multiplicities.

$$\mathcal{L}(M) = (\ell_1, \ell_2, \ldots), \quad \text{where} \quad \ell_i \leq \ell_{i+1}.$$

$\mathcal{L}(M)$ is closely related to $\mathcal{E}(M)$, the Laplace eigenvalue spectrum.

- For closed manifolds, $\mathcal{L}(M)$ determines $\mathcal{E}(M)$. [Kelmer, 2011]
- $\mathcal{E}(M)$ determines the length set $L(M) = \{\ell_i\}$, without multiplicities. [Duistermaat–Guillemin, 1975]

Highly abridged history

Question: Can you hear the shape of a drum?

Answer: No! There are lots of examples of hyperbolic manifolds that are *isospectral* (same length spectrum) but not isometric.

- 1980: Vignéras gives arithmetic construction.
- 1985: Sunada develops general method.

Theorem (Sunada)

Let M be a non-positively curved manifold, and $\varphi : \pi_1 M \to G$ a map onto a finite group. Suppose that $H, K \subset G$ are almost conjugate, meaning:

$$\forall g \in G, \qquad \#\big([g] \cap H\big) = \#\big([g] \cap K\big).$$

Then M_H and M_K, covers corresponding to $\varphi^{-1} H$ and $\varphi^{-1} K$, are isospectral.

- 2006: McReynolds uses Sunada's construction to build large families of hyperbolic 3–manifolds that are isospectral but not isometric. Size of family is super-polynomial in volume.

Hearing shape of a manifold, up to commensurability

Question (Reid)**:** If $\mathcal{L}(M_1) = \mathcal{L}(M_2)$, must M_1 and M_2 be *commensurable*? That is, must they share a finite-sheeted cover?

- Yes in all the above examples.
- (Reid, 1992): Yes if the M_i are *arithmetic* surfaces. Think $\mathbb{H}^2/SL(2,\mathbb{Z})$.
- (Chinburg–Hamilton–Long–Reid, 2007): Yes if the M_i are arithmetic hyperbolic 3–manifolds. Think $\mathbb{H}^3/SL(2,\mathbb{Z}[\sqrt{-d}])$.
- (Prasad–Rapinchuk, 2008): Yes if the M_i are arithmetic hyperbolic d–manifolds, where $d \neq 1 \mod 4$.

Theorem (Linowitz–McReynolds–Pollack–Thompson, 2014)

Let M_1 and M_2 be arithmetic hyperbolic 3–manifolds with $Vol(M_i) \leq V$. If

$$\mathcal{L}(M_1)|_{\leq A} = \mathcal{L}(M_1)|_{\leq A}, \qquad \textit{where} \qquad A = c \cdot \exp\left((\log V)^{\log V}\right) > \exp(V^k)$$

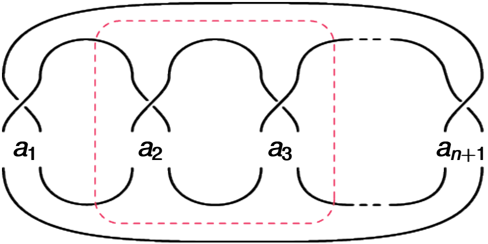
then M_1 and M_2 are commensurable. There is a similar result in dimension 2.

Geometrically similar pretzel knots

Theorem (Millichap, 2014)

For each (odd) $n \geq 5$, there are $n!/2$ distinct hyperbolic knot complements $\{M_n^\sigma\}$, where $\sigma \in S_n$, such that

- *$Vol(M_n^\sigma) = Vol(M_n^{\sigma'}) = v_n \approx n \cdot Vol(\text{ideal octahedron})$.*
- *Each M_n^σ is the only knot complement in its commensurability class.*
- *Each M_n^σ has the same $(n+1)$ shortest geodesics.*



Construction: Pretzel knot $P(a_1, \ldots, a_{n+1})$, modified via *mutation* along Conway spheres.

The catch: the $(n+1)$ shortest geodesics have length $\ell_i \leq 0.015$.

Geometrically similar but not commensurable

Theorem (F–Millichap, 2016)

For each $n \gg 0$, there are distinct hyperbolic 3–manifolds M_n and M'_n s.t.:

1. $Vol(M_n) = Vol(M'_n) \approx n$. *[$\approx$: equal up to multiplicative constants]*
2. *M_n (also, M'_n) is minimal in its commensurability class. So the two are not commensurable.*
3. *Length spectra agree up to length n:* $\mathcal{L}(M_n)|_{\leq n} = \mathcal{L}(M'_n)|_{\leq n}$.
4. *M_n and M'_n have at least e^n/n geodesics up to length n.*

Remarks and variations:

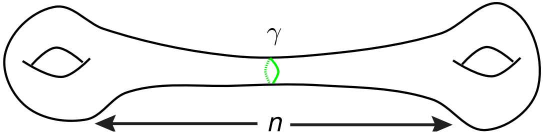
- M_n and M'_n can be taken closed, or non-compact with finite volume.
- We can take $M_n = S^3 \setminus K_n$ and $M'_n = \setminus K'_n$ to be knot complements in S^3, at the cost of making the length cutoff $A = 2 \log n$.
- The length cutoff in the LMPT theorem is much higher: $A > \exp(V^k)$.

Interlude: geometrically similar surfaces

Observation

Let S be a surface of Euler characteristic $\chi(S) < -1$. Then, for all $n > 0$, S admits a pair of complete hyperbolic structures Σ_n and Σ_n' such that:

1. $Area(\Sigma_n) = Area(\Sigma_n') = -2\pi\chi(S)$.
2. *Length spectra agree up to length n:* $\mathcal{L}(\Sigma_n)|_{\leq n} = \mathcal{L}(\Sigma_n')|_{\leq n}$.
3. Σ_n *and* Σ_n' *are not commensurable.*



Construction:

- Choose metric Σ_n where γ has collar of width $> n$.
- To get Σ_n': cut along γ, twist, re-glue.

The construction (for knots)

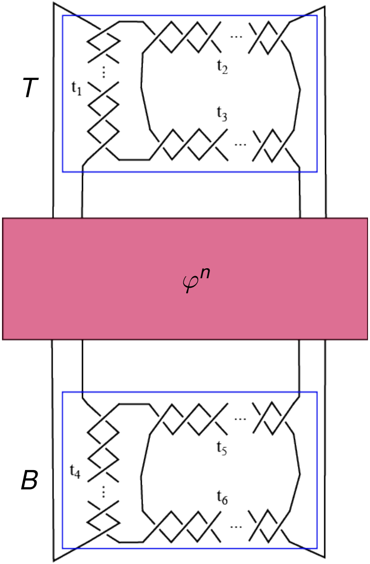
Knots with these properties are abundant, if you know where to look.

To build K_n, start with a pair of tangles T (top) and B (bottom) that have:

- Incompressible boundary.
- No symmetries.

Connect them with a long pure braid φ^n, where $\varphi \in \mathrm{Mod}\,(S_{0,4})$ is pseudo-Anosov.

To build K_n', use tangle B', namely B rotated 180°. The two knots are *mutants*.



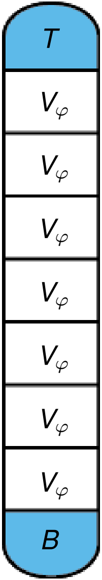
Geometry of $M_n = S^3 \setminus K_n$

As $n \to \infty$, M_n looks more and more like this:

- Caps corresponding to T and B
- Caps separated by $\approx n$ copies of a submanifold *nearly isometric* to V_φ, where V_φ is a fundamental domain for M_φ, the mapping torus of φ.
- $\mathrm{Vol}\,(M_n) \approx n \cdot \mathrm{Vol}\,(V_\varphi) \approx n$.
- T and B separated by collar of width $> n$.

The geometric limit as $n \to \infty$ was first described by Namazi–Souto.

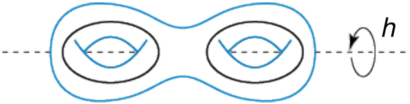
Width of collar follows by work of Brock–Bromberg, Minsky, and Bowditch.



Geometry of closed manifolds M_n

Closed manifolds M_n and M'_n are built in a very analogous way. Now, caps T and B have boundary a genus 2 surface.

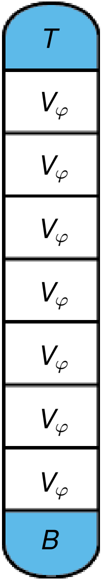
- For M_n: glue T to B by φ^n.
- For M'_n: glue T to B by $\varphi^n \circ h$.



Key: cutting & regluing by h is a *rigid* process.

Lemma

Any closed geodesic $\gamma \subset M_n$ that is not homotopic into T or B must have length at least n. Thus all geodesics shorter than n remain invariant when we mutate M_n to M'_n.



Ruling out commensurability

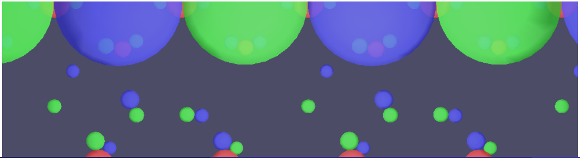
Theorem (Margulis)

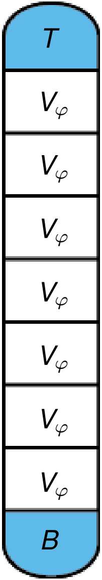
A hyperbolic manifold M with $Vol(M) < \infty$ is non-arithmetic $\Leftrightarrow$ the commensurability class of M has a unique minimal element.

M_n and M_n' are not arithmetic. Thus they are commensurable $\Leftrightarrow$ they cover a common orbifold quotient, $\mathcal{O}$.

Regular covers come from *symmetries*. But we know the geometry of M_n, and it is highly asymmetric.

Irregular covers come from *hidden symmetries*. These are ruled out by a delicate argument using pants on ∂T, Dehn filling, and horoball packings of $\mathbb{H}^3$.





Counting geodesics in M_n

Theorem (Huber, Margulis)

Let M be a finite volume hyperbolic $(d+1)$–manifold. The number of closed geodesics in M of length $\leq L$ is $\pi_M(L) \sim \dfrac{e^{dL}}{dL}$. *[$\sim$: ratio $\to 1$].*

The limit does not depend on M! But the rate of convergence does.

Problem: We want a *uniform* result that will work in every M_n for $n \gg 0$.

Solution: Count closed geodesics in a *pleated surface* separating two consecutive copies of the block V_φ. The geometry of these surfaces converges.