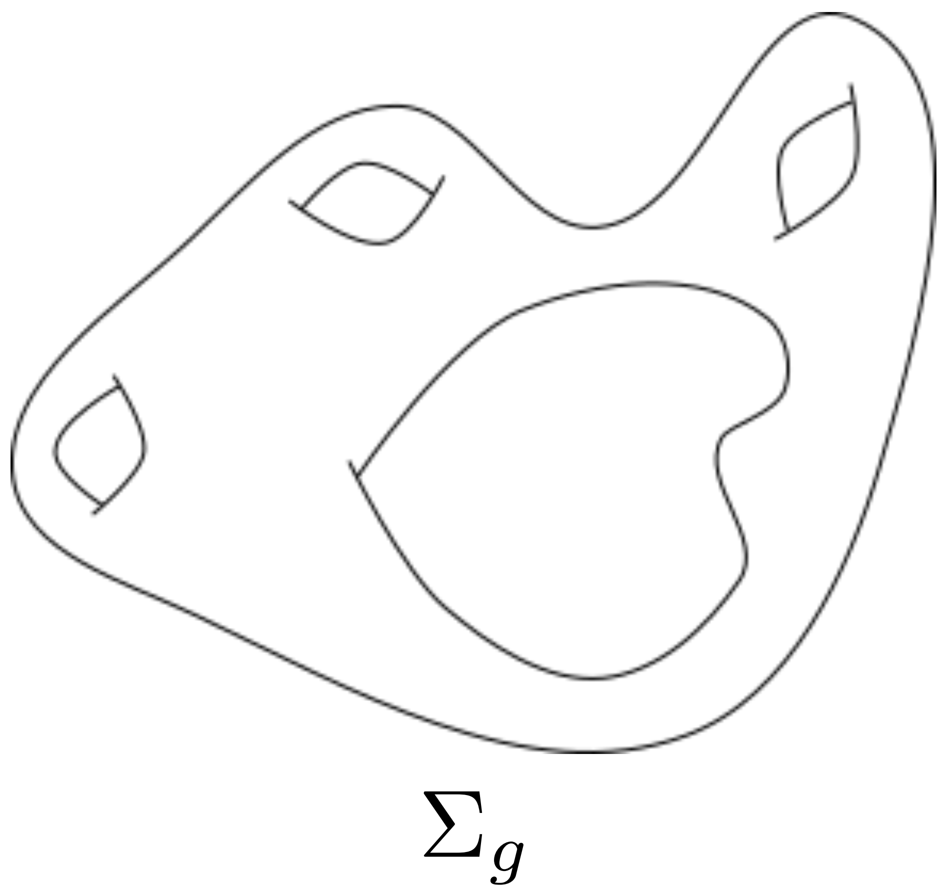
Lightning Talks I
Tech Topology Conference

December 10, 2016

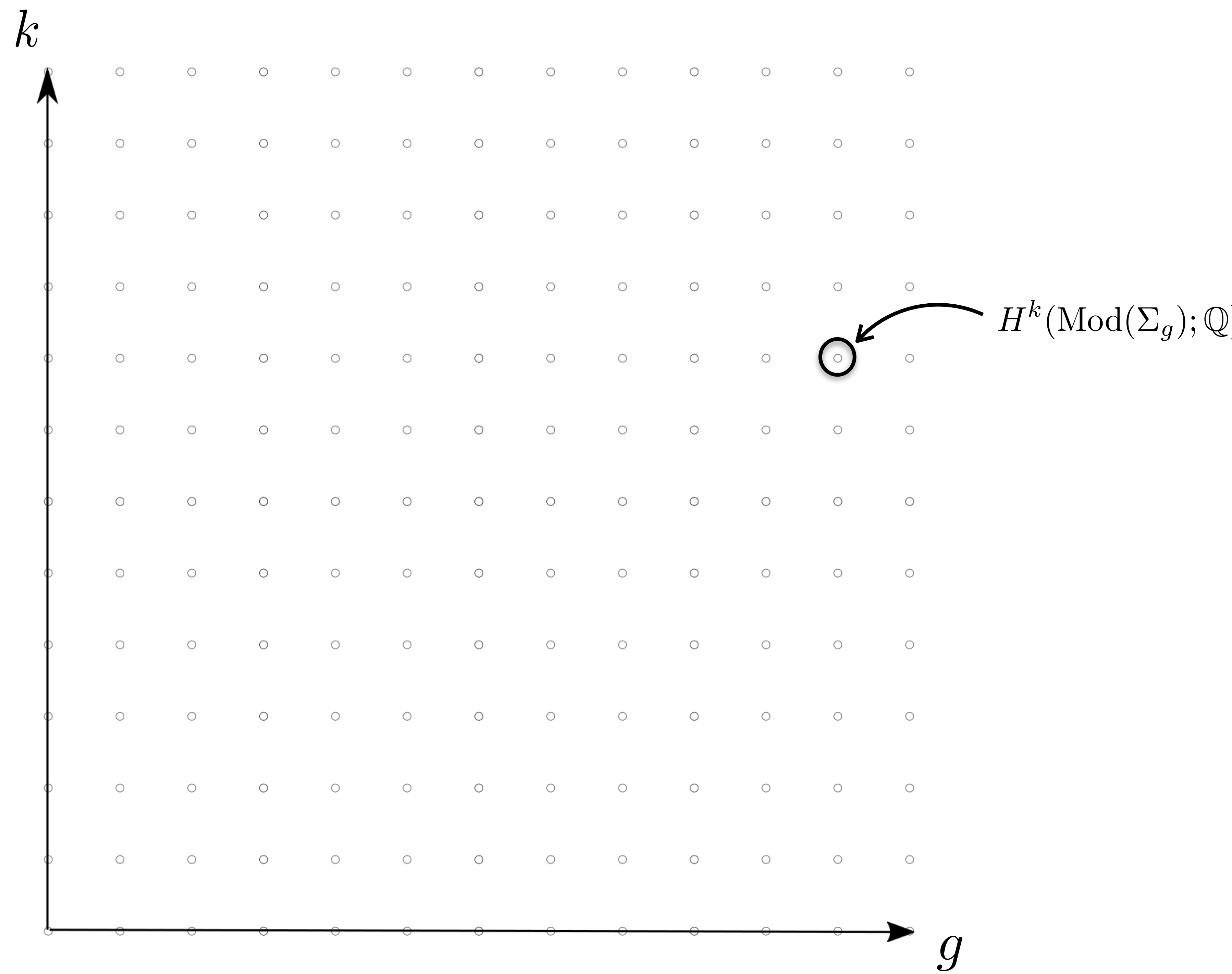
Top-dimensional cohomology in the mapping class group



$$\mathrm{Mod}(\Sigma_g) := \mathrm{Homeo}^+(\Sigma_g)/\sim$$

Neil J. Fullarton
Rice University
(joint with Andrew Putman)

$H^*(\mathrm{Mod}(\Sigma_g); \mathbb{Q})$: the state of play



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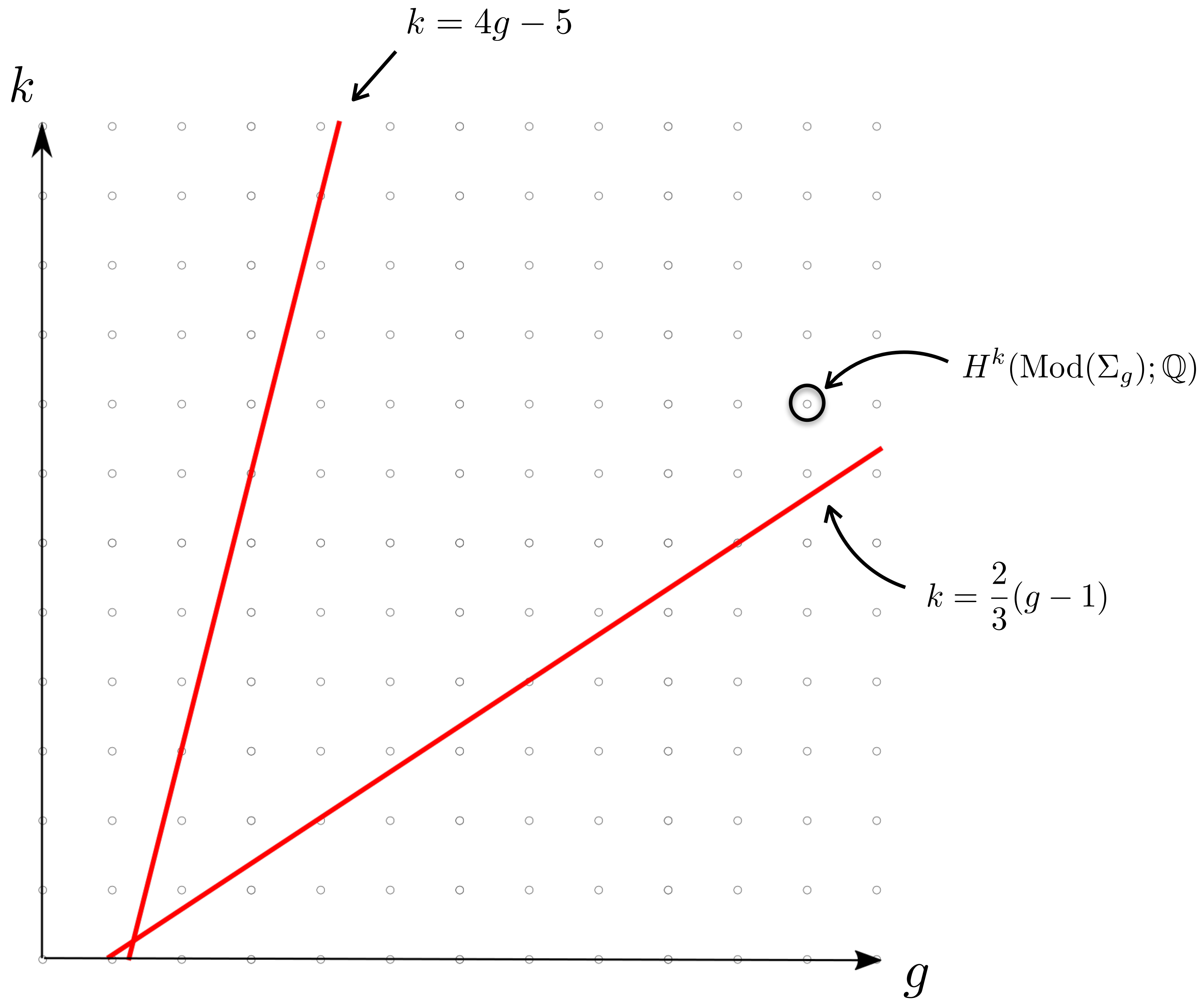

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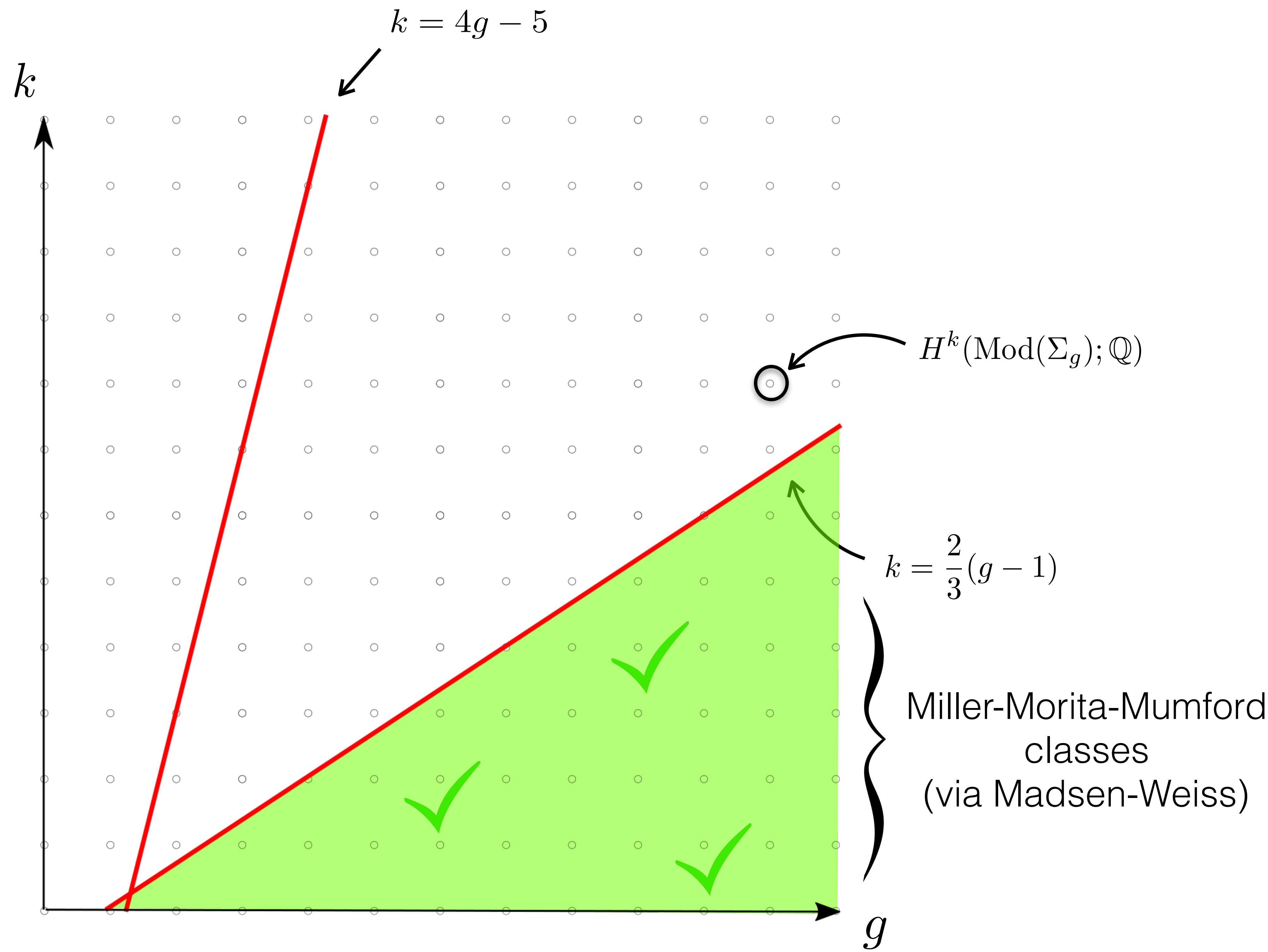

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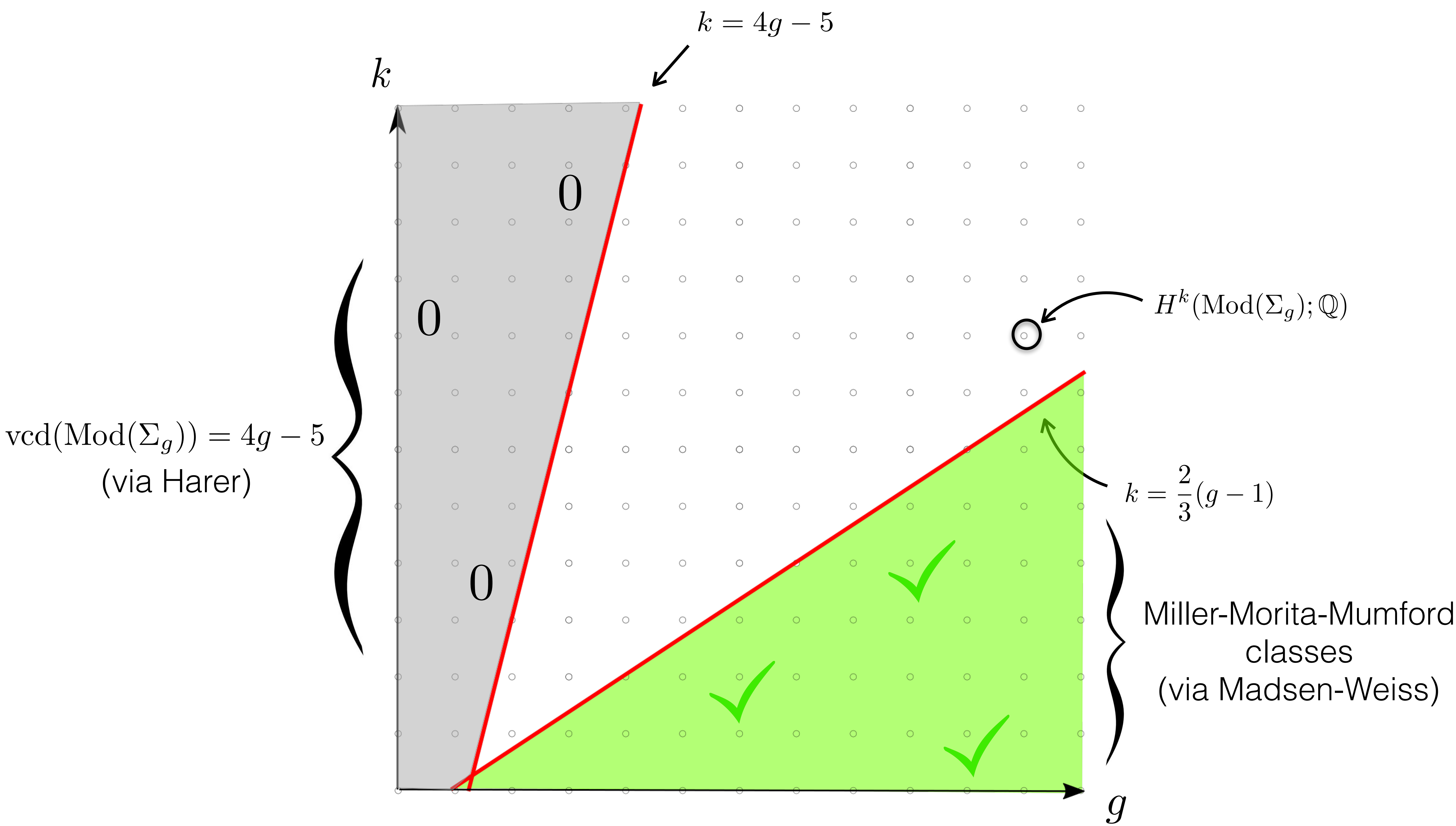

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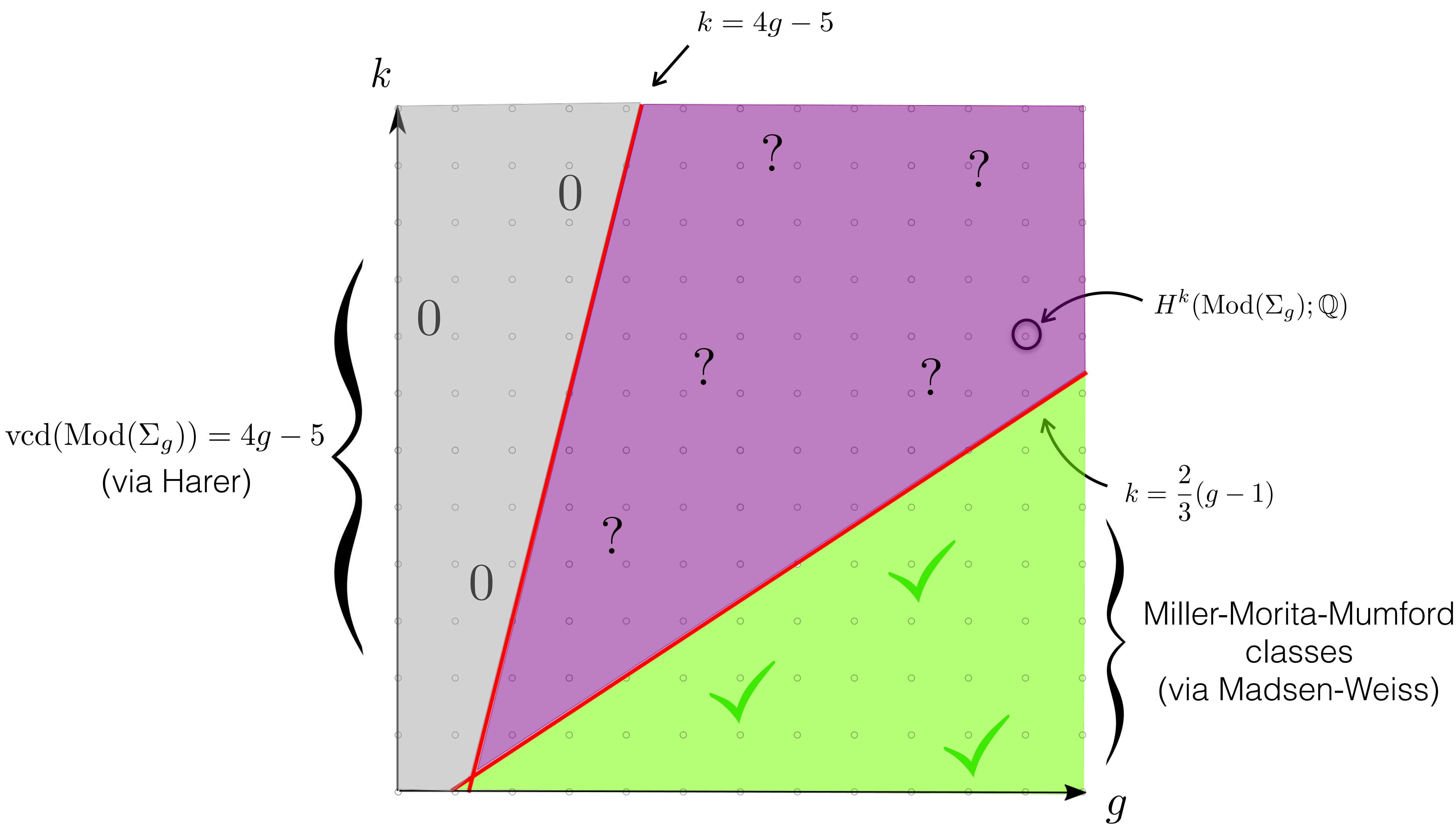

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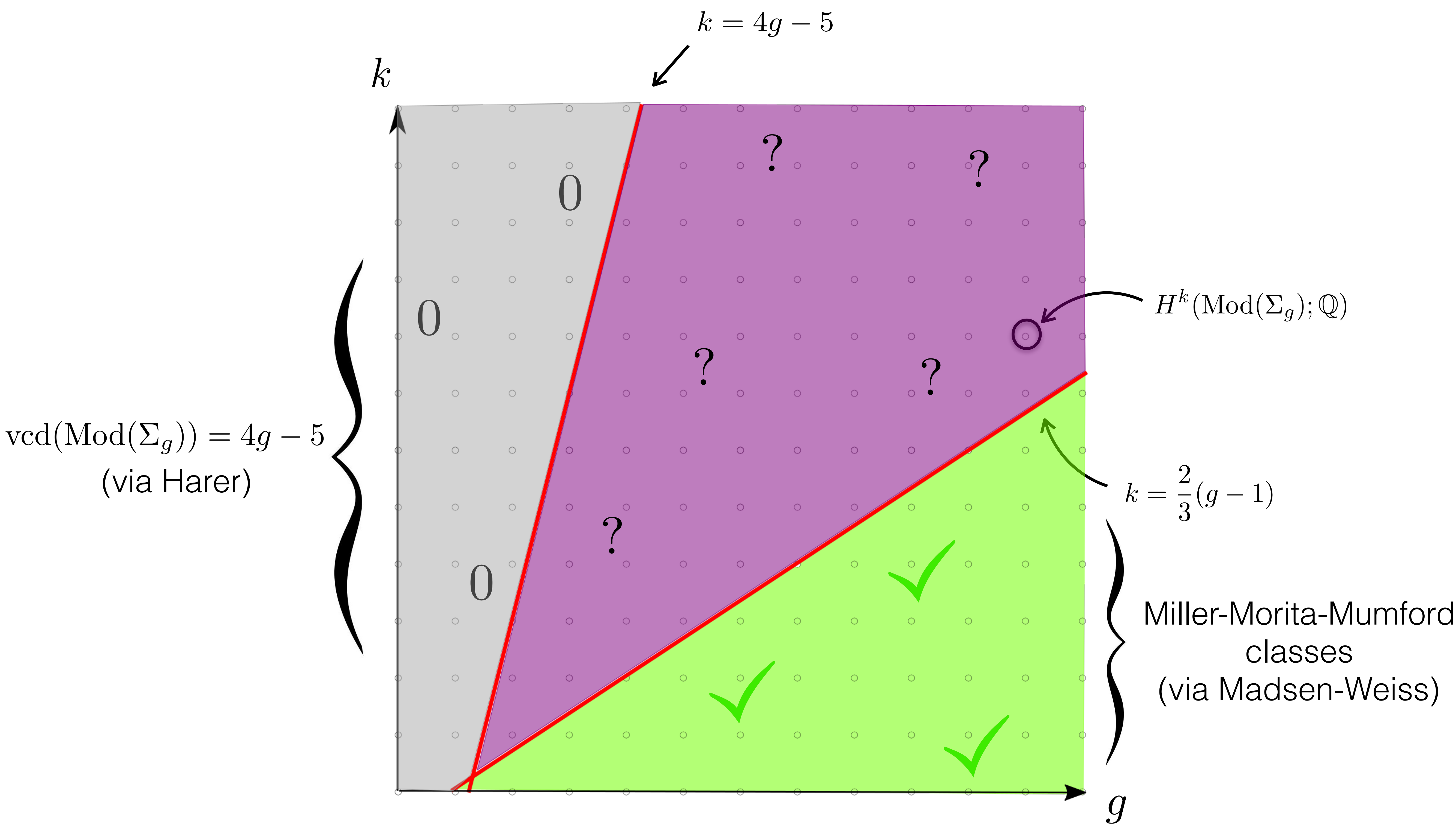

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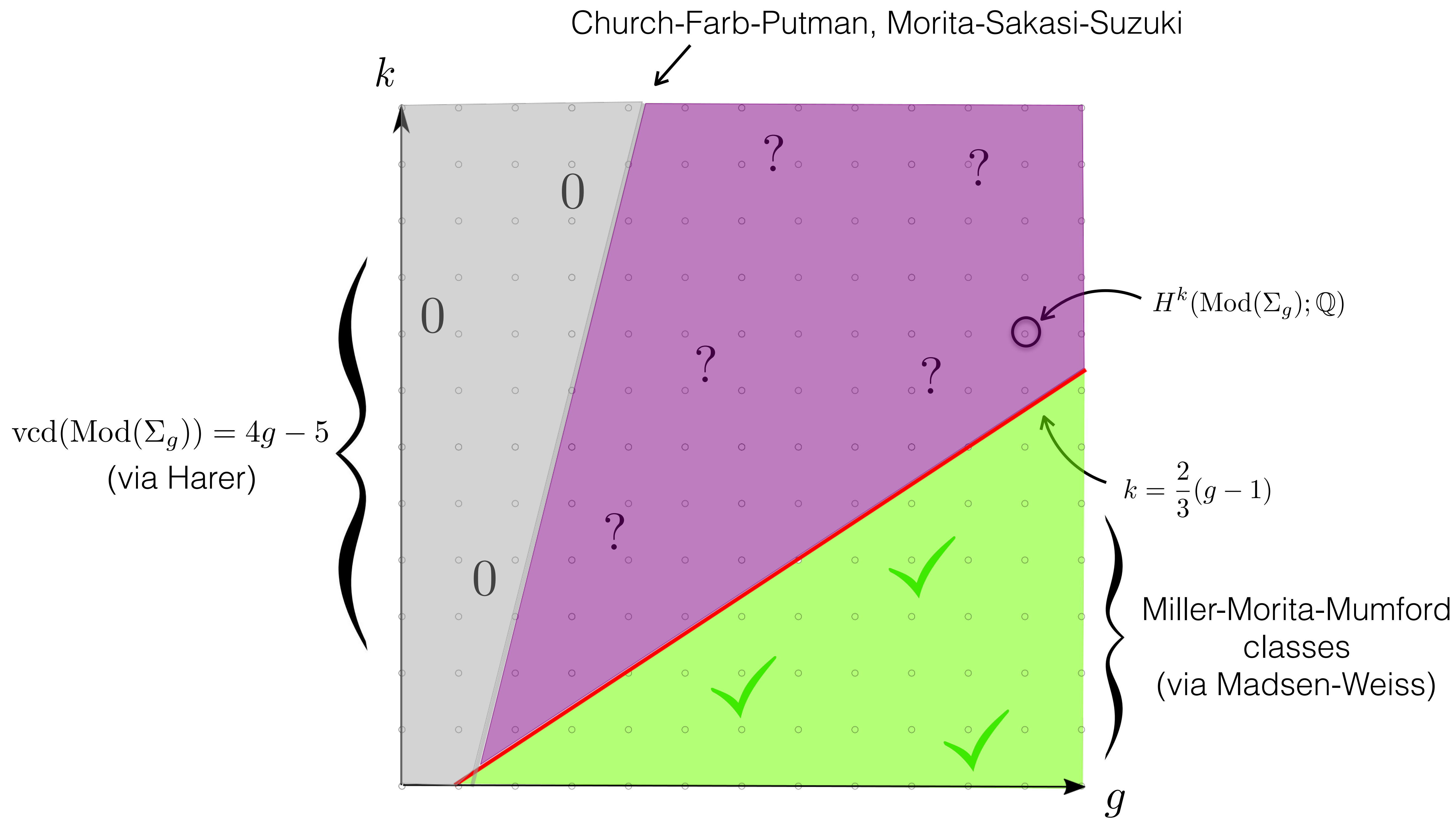

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Theorem (F-Putman). Let $g, \ell \geq 2$ and $p \mid \ell$ be prime. Then

$$\dim_{\mathbb{Q}} H^{4g-5}(\mathrm{Mod}(\Sigma_g, \ell); \mathbb{Q}) \geq \frac{|\mathrm{Sp}_{2g}(\mathbb{Z}/p)|}{g(p^{2g}-1)}.$$

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Corollary. The *coherent cohomological dimension* of moduli space is at least $g-2$.

Ideas behind Theorem’s proof

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(1) Duality:

$$H^{4g-5}(\mathrm{Mod}(\Sigma_g, \ell); \mathbb{Q}) \cong H_0(\mathrm{Mod}(\Sigma_g, \ell); \mathrm{St}(\Sigma_g))$$

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Recall:

$$H_0(G; M) = M / \langle m - g \cdot m \mid m \in M, g \in G \rangle$$

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Takeaway: must understand the action

$$\mathrm{Mod}(\Sigma_g, \ell) \circlearrowright \mathrm{St}(\Sigma_g)$$

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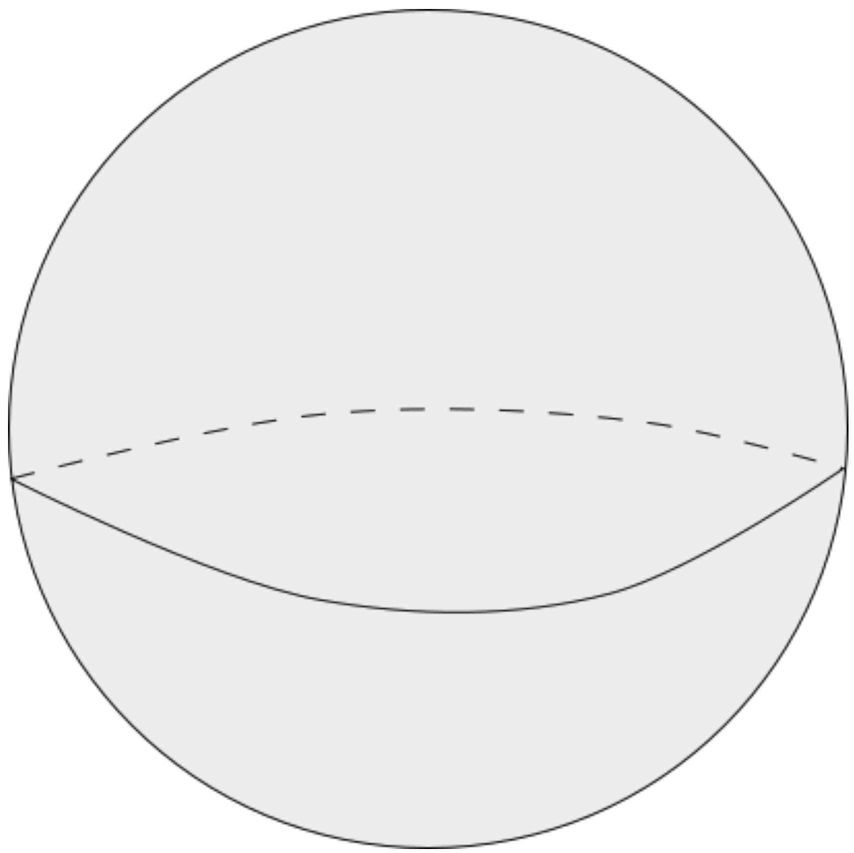

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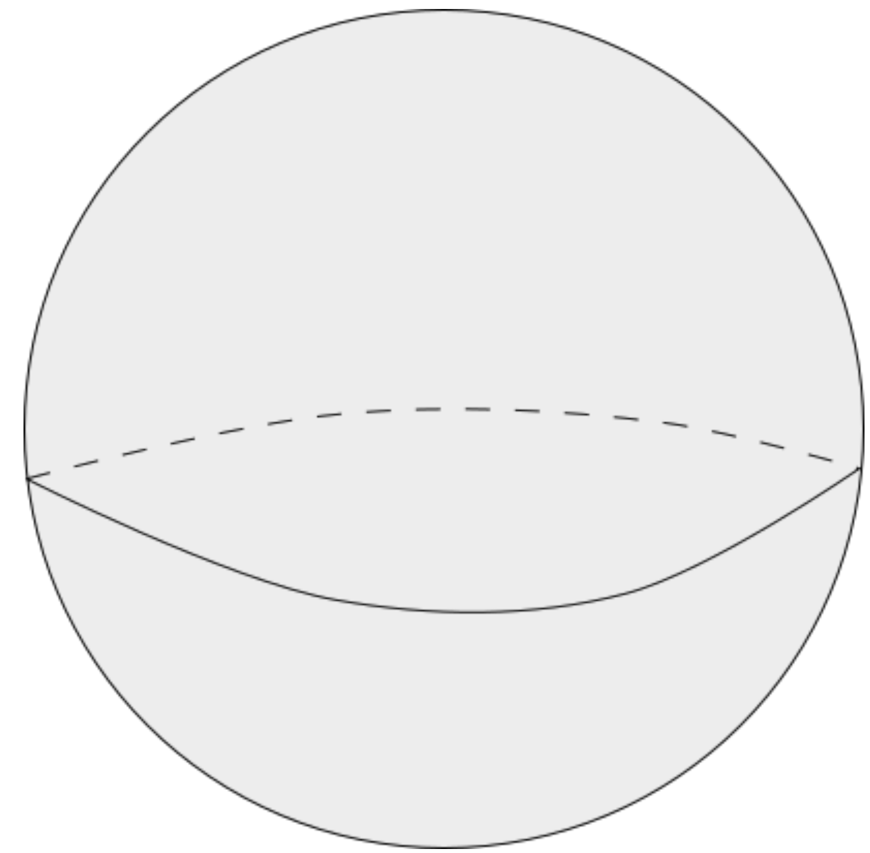

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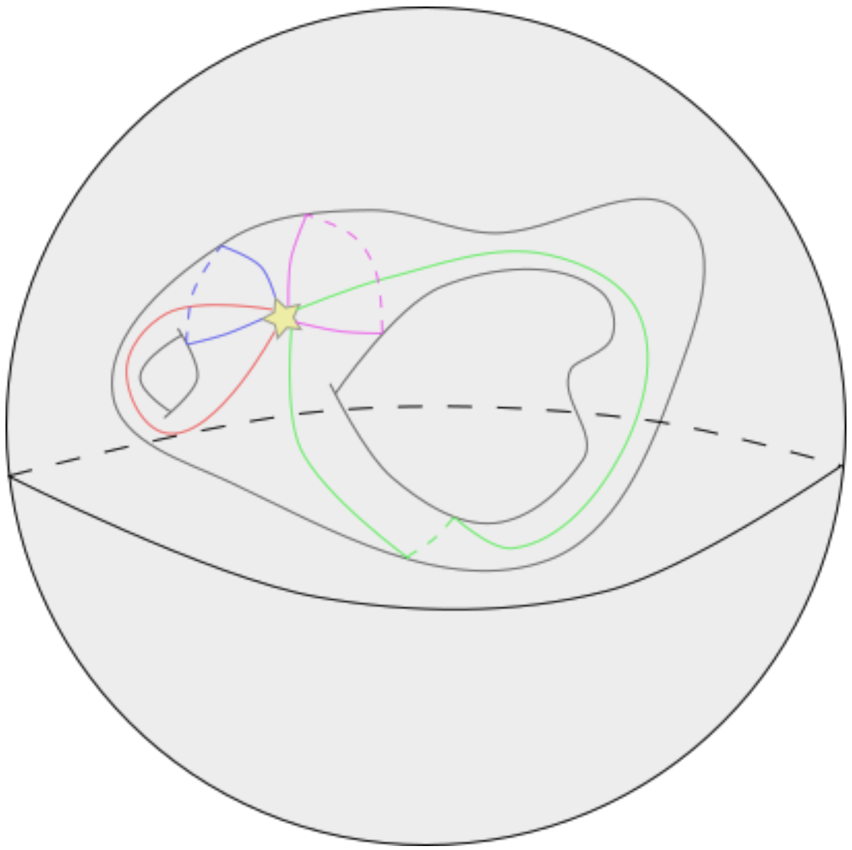

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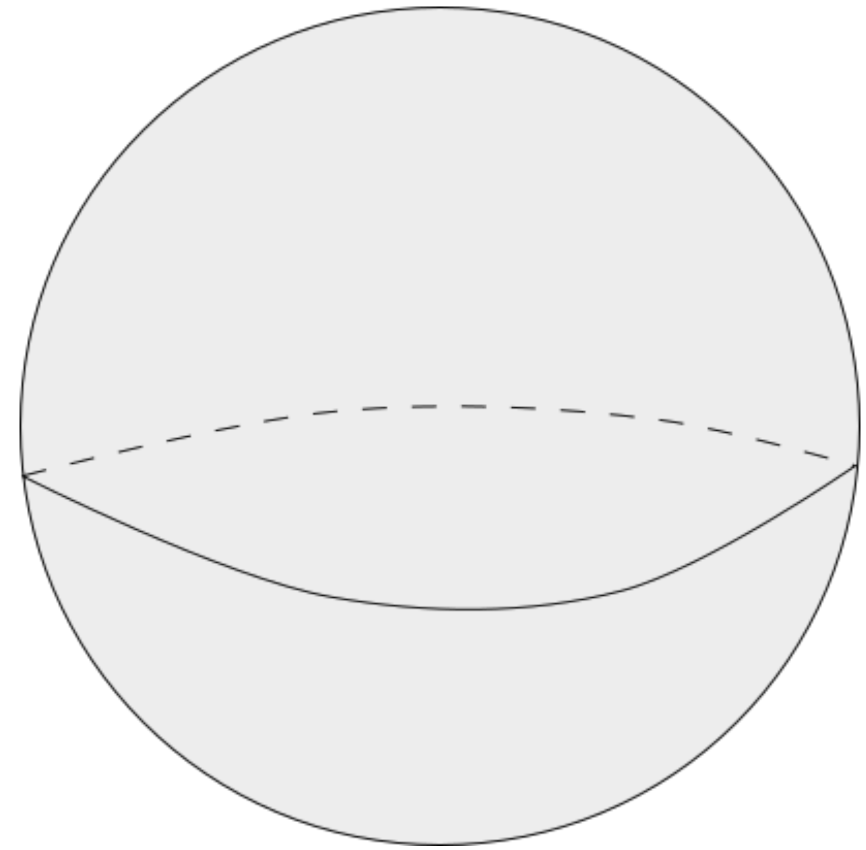

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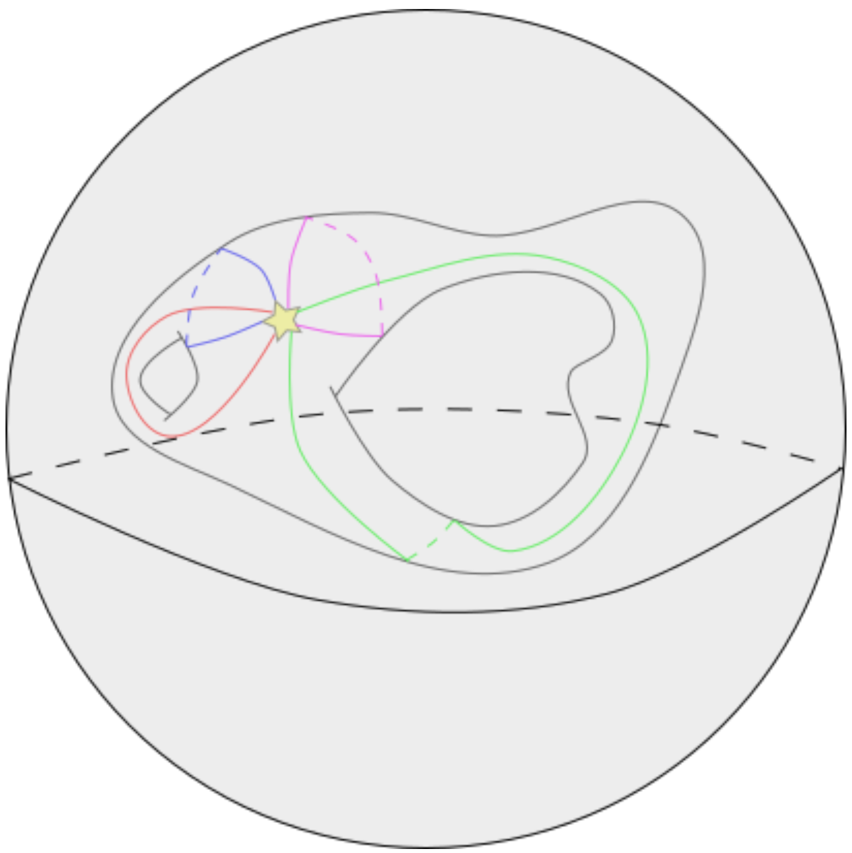

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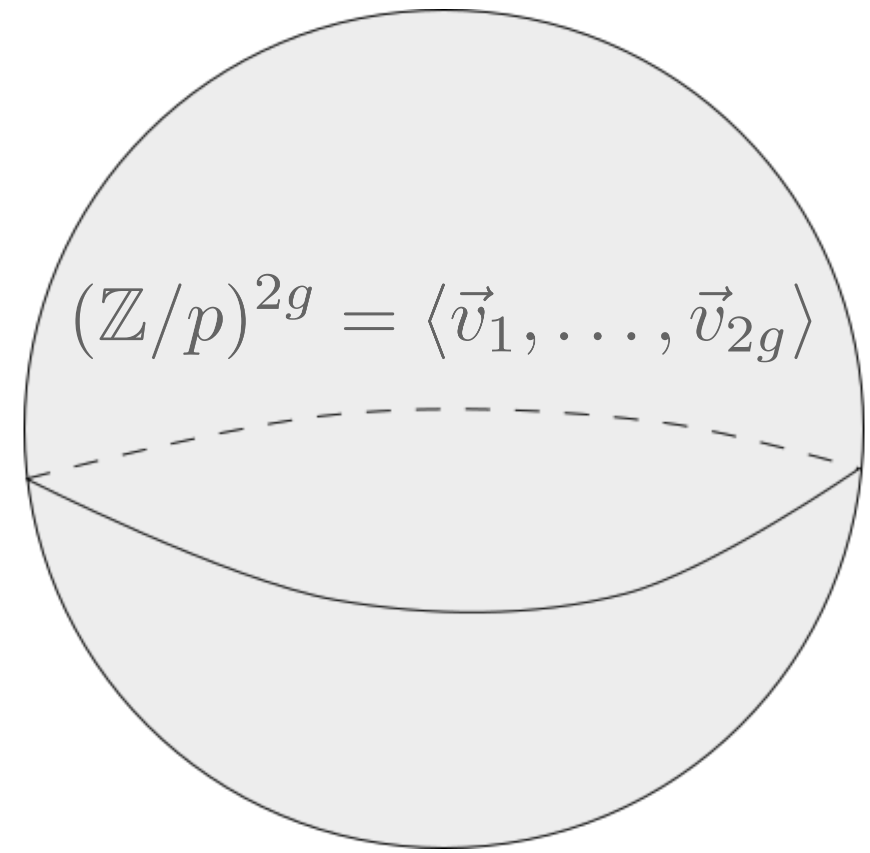

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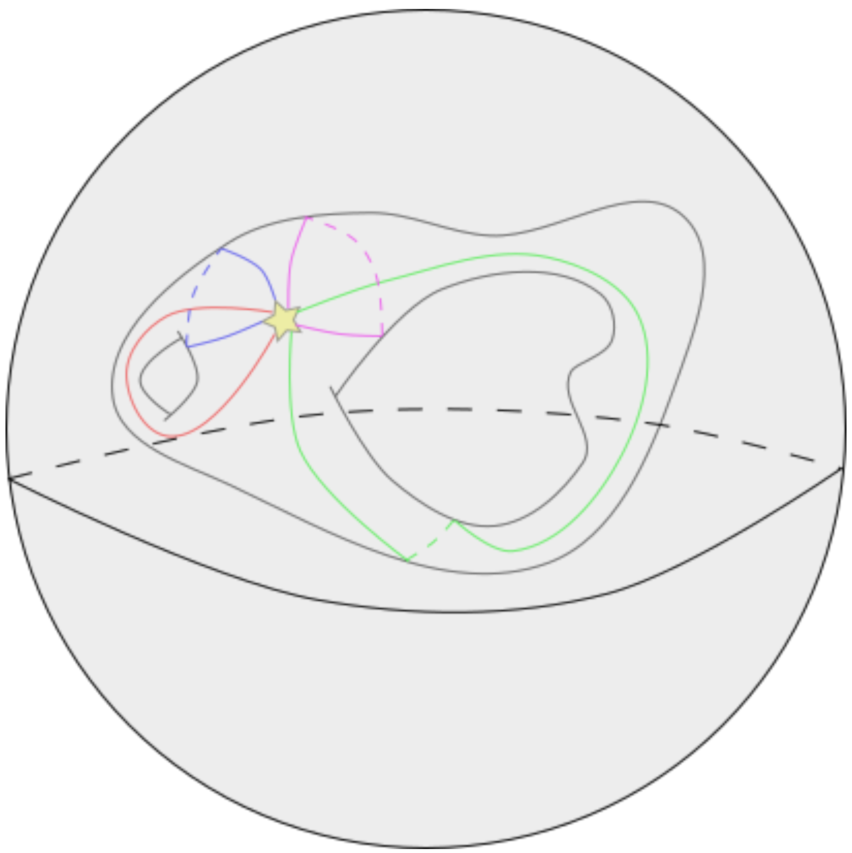

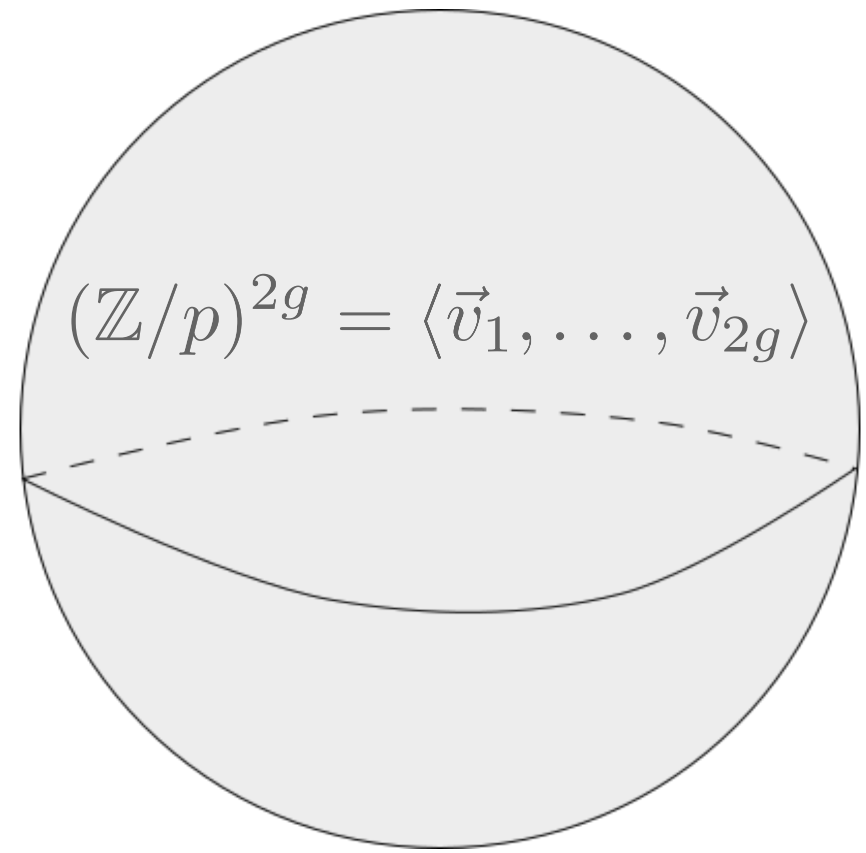

$$\mathrm{St}(\Sigma_g) \xrightarrow{H_1(-;\mathbb{Z}/p)} \mathrm{St}_{2g}(\mathbb{Z}/p)$$

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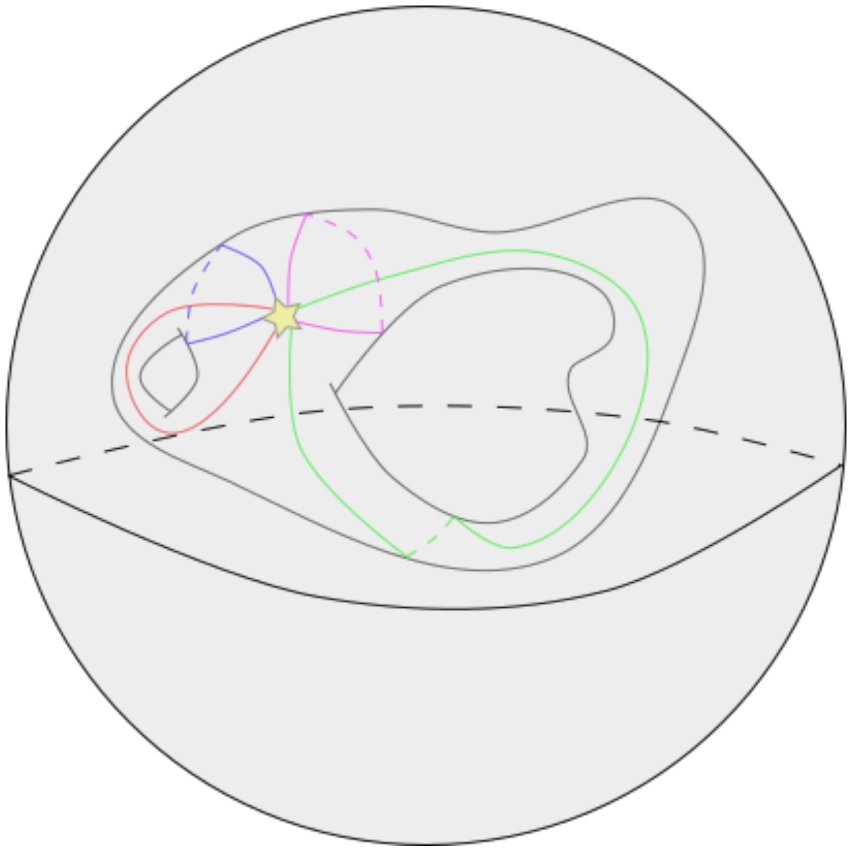

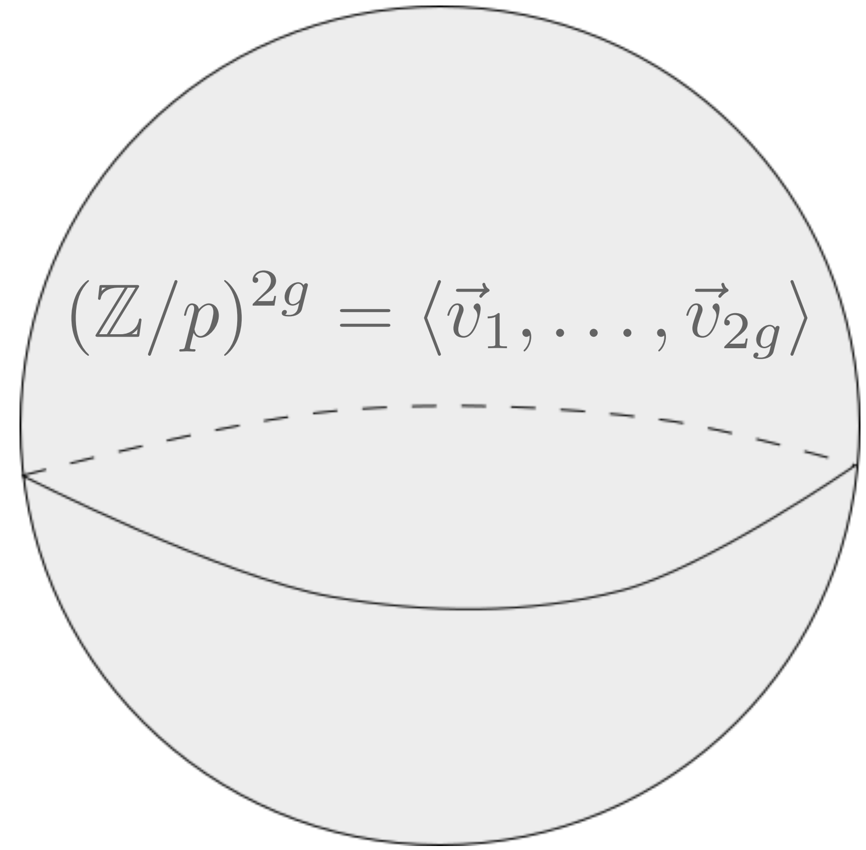

$\mathrm{St}(\Sigma_g)$ (via the curve cx.) $\xrightarrow{H_1(-;\mathbb{Z}/p)}$ $\mathrm{St}_{2g}(\mathbb{Z}/p)$ (via *Tits building*)

$\mathrm{St}(\Sigma_g) \to H_0(\mathrm{Mod}(\Sigma_g, \ell); \mathrm{St}(\Sigma_g)) \to \mathrm{St}_{2g}(\mathbb{Z}/p)$

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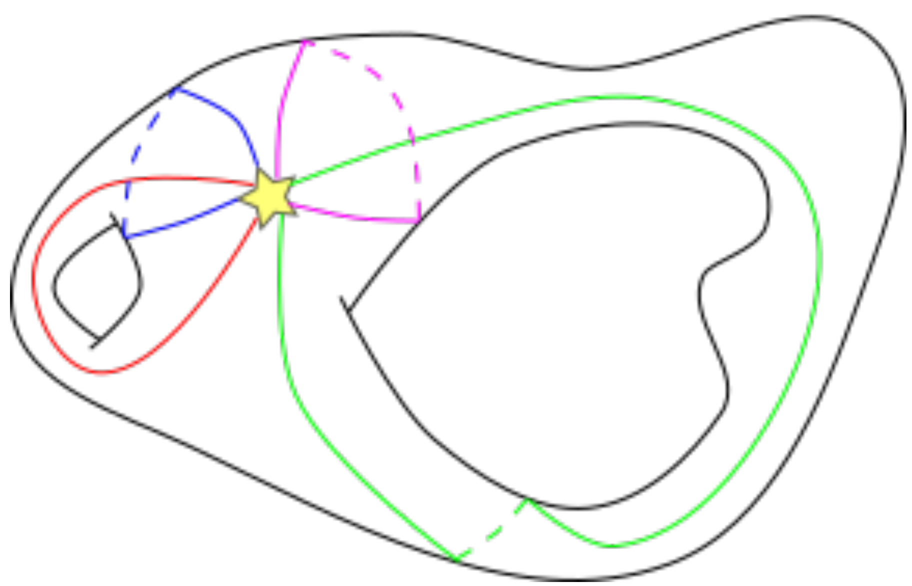

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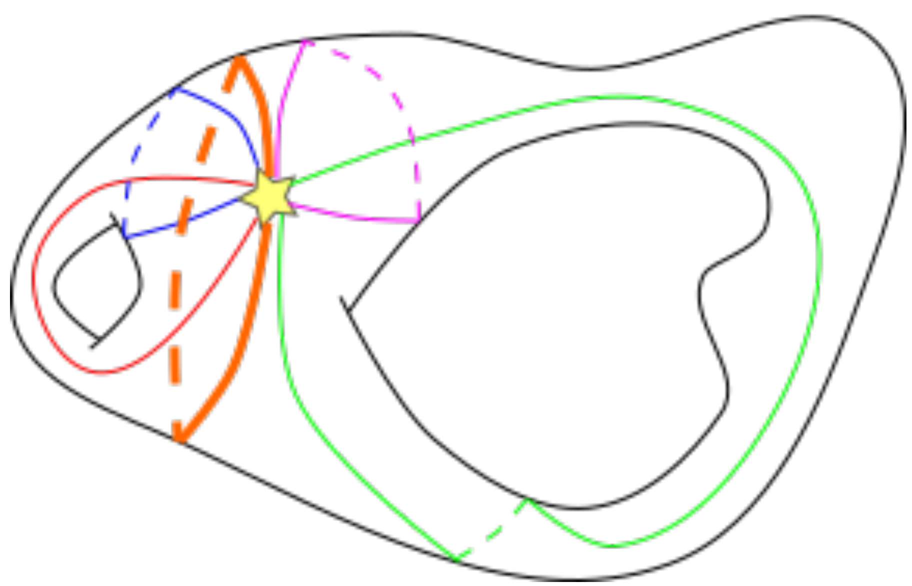

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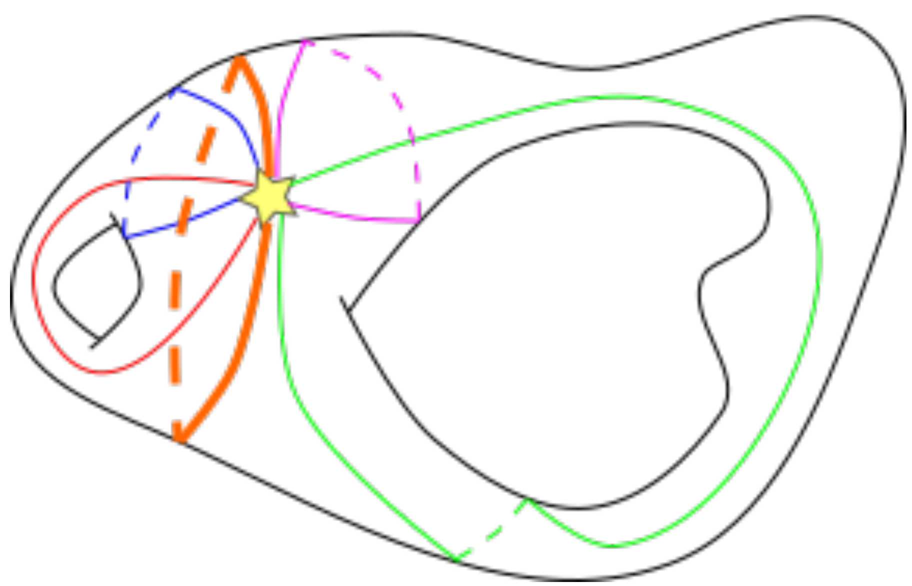

$$\longrightarrow \quad (\mathbb{Z}/p)^4 = L \oplus R$$

Kill the span $\mathcal{SB}$ of such 'separated' bases:

$$\mathrm{St}(\Sigma_g) \xrightarrow{H_1(-;\mathbb{Z}/p)} \mathrm{St}_{2g}(\mathbb{Z}/p)/\mathcal{SB}$$

$$\mathrm{St}(\Sigma_g) \searrow H_0(\mathrm{Mod}(\Sigma_g, \ell); \mathrm{St}(\Sigma_g)) \nearrow \mathrm{St}_{2g}(\mathbb{Z}/p)/\mathcal{SB}$$

Picard groups of moduli spaces of Riemann surfaces with symmetry

Kevin Kordek
Texas A&M University

The problem

- Let $g \geq 2$ and suppose $H < \mathrm{Mod}(S_g)$ is a finite subgroup.
- (Nielsen Realization) H lifts to a group of automorphisms of some Riemann surface structure on S_g.

Problem:

Investigate the structure of the moduli space M_g^H of genus g Riemann surfaces with a group of automorphisms acting topologically like H.

The moduli space

Theorem (González-Díez + Harvey)

$M_g^H = Teich_g^H / Mod_H(S_g)$

- Teich_g^H is the fixed locus of H in Teichmüller space Teich_g (contractible complex submanifold!)
- $\text{Mod}_H(S_g)$ is the normalizer of H in $\text{Mod}(S_g)$.

Observation 1: M_g^H is a quotient of a smooth complex quasiprojective variety by a finite group (a quasiprojective orbifold).

Observation 2: M_g^H has the same rational cohomology as $\text{Mod}_H(S_g)$.

Picard groups

The Picard group is an algebro-geometric invariant:

$$\text{Pic } M_g^H = \{\text{isomorphism classes of algebraic line bundles on } M_g^H\}$$

(Zariski-locally trivial, algebraic transition functions).

Theorem (K.)

Suppose $H < Mod(S_g)$ is finite+abelian. Let $g' =$ genus of S_g/H.

1. *If $g' = 0$, then Pic M_g^H is finite.*
2. *If $g' \geq 3$, then Pic M_g^H is finitely generated.*

The proof

Idea of proof of Part 2:

- Show the (rational) first Chern class

$$c_1 : \mathrm{Pic}\ M_g^H \otimes \mathbb{Q} \to H^2(M_g^H, \mathbb{Q})$$

  is injective.

- Comes down to showing that

$$H^1(M_g^H, \mathbb{Q}) \cong H^1(\mathrm{Mod}_H(S_g), \mathbb{Q}) = 0.$$

The proof

- (Birman-Hilden, Harvey-MacLachlan)

$$\implies \mathrm{Mod}_H(S_g)/H \;\cong\; \text{finite-index } \Gamma < \mathrm{Mod}(S_{h,n})$$

where $n = \#\,\{\text{branch points of } S_g \to S_g/H\}$.

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H abelian $\implies$ Γ contains all Dehn twists on separating curves.

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- A theorem of Putman + fiddling $\implies$ $H^1(\Gamma, \mathbb{Q}) = 0$.

- H finite $\implies$ $H^1(\text{Mod}_H(S_g), \mathbb{Q}) = 0$.

Future work

Some questions:

- What happens when $h = 1, 2$?
- What if H is non-abelian?

Thank you!

Sutured Khovanov Homology and Tight Links

I. Banfield[1]

[1]Department of Mathematics
Boston College

Tech Topology Conference, 2016

Outline

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1. Motivation
   - What is the contact-geometric information contained in Khovanov homology?
2. Strongly Quasipositive and Tight Links
3. Staircases
4. A Conjecture

Knot Homology Theories and Contact Structures

- (Hedden, Rudolph, 2007) Knot Floer homology detects membership in the class of links inducing the tight contact structure on S^3.
- Is a similar statement true for Khovanov homology?

Khovanov Chain Complex

- **Generators**: Smoothings of a link diagram.
- **Maps**: measure the behavior of smoothings under a change of the resolution of a crossing.
- For braid diagrams, get a filtration by singular homology class of the generators. The associated graded complex is the **sutured Khovanov complex**.

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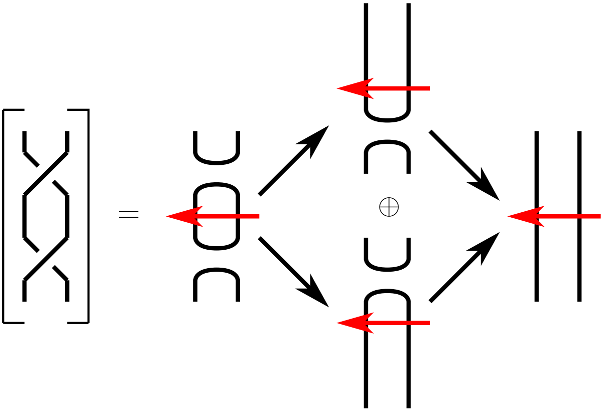
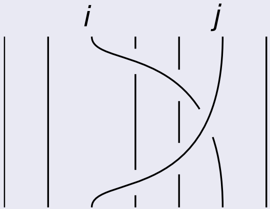
(Sutured) Khovanov Chain complex - Picture



Band Generators

Definition

Let B_n be the braid group on n stands. The elements

$a_{j,i} =$ 

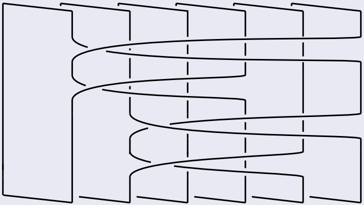
are called **band generators** and generate the braid group B_n.

Strongly Quasipositive Links and Tight Links

Definition (Rudolph)

A link $L \subset S^3$ is **strongly quasipositive** if it admits a braid representative which contains positive band generators only.

Example $\beta = a_{1,6}a_{1,4}a_{2,6}a_{2,5}$.

$L = \hat{\beta} =$ 

Tight Links

Theorem (Giroux, Rudolph)

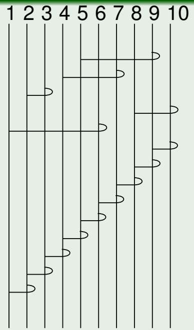
*The fibered links inducing the tight contact structure on S^3 are exactly the fibered strongly quasipositive links. Such a link is called **tight**.*

Staircase Braid Closures

Definition (B.)

A **staircase braid** is a strongly quasipositive braid $\beta \in B_n$ which contains the Dual Garside element $\delta = \sigma_{n-1}\sigma_{n-2}\dots\sigma_1$.

Example



Properties of Staircases

Theorem (B.)

Staircase braid closures are fibered and so are tight. Further, the monodromy is a product of Dehn twists.

Theorem (B. - Rudolph)

Closures of positive braids are staircase braid closures. Conversely, staircase braid closures are stably positive braid closures.

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All inclusions are proper:

$$\{\textit{positive braids}\} \subset \{\textit{staircase braids}\} \subset \{\textit{tight links}\} \qquad (1)$$

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Conjecture

A braid closure $\hat{\beta} \subset S^3$ is tight if and only if the sutured Khovanov homology of $\hat{\beta}$ is

$$SKh_i(\hat{\beta}) = \begin{cases} 0 & \text{if } i < 0 \\ V^n & \text{if } i = 0 \\ V^{n-2} & \text{if } i = 1 \\ \star & \text{if } i > 1. \end{cases} \tag{2}$$

Thank you for listening!

Stein fillings of Legendrian surgeries with enough stabilizations

Alex Moody

University of Texas at Austin

December 10,2016

Background

Definition

A contact 3-manifold is (for the purposes of this talk) a closed orientable 3-manifold Y equipped with a two dimensional coorientable subbundle ξ of TY satisfying a nonintegrability condition (locally looks like $\alpha = 0$ for some 1-form α with $\alpha \wedge d\alpha > 0$).

Definition

A symplectic filling of (Y, ξ) is a compact symplectic 4-manifold (X, ω) with boundary Y where ξ is the complex tangencies for a nice (compatible) almost complex structure, and a little more structure (a Liouville vector field near the boundary).

Stein Fillings

A Stein filling is a particular kind of symplectic filling.

Example

The unit 4-ball B^4 in $\mathbb{C}^2$ is a Stein filling of (S^3, ξ_{std}).

Example

If $f : \mathbb{C}^3 \to \mathbb{C}$ is a complex polynomial and 0 is a regular value of f. Then $f^{-1}(0) \cap B^6$ is a Stein filling of $f^{-1}(0) \cap S^5$ for some large enough round B^6. For instance if we let $f(x, y, z) = x^2 + y^3 + z^5 - 1$ we get a Stein filling of the Poincare homology sphere.

Central Questions

Question (Classification)

Given (Y, ξ) a contact 3-manifold. What are all the Stein fillings (X, ω) of (Y, ξ) up to symplectic deformation, symplectomorphism or diffeomorphism?

Question (Geography)

Given (Y, ξ) a contact 3-manifold. What are the possible values for $\chi(X)$ and $\sigma(X)$ for (X, ω) a Stein filling of (Y, ξ)?

Some Known Results

Symplectic fillings can often be completely classified in the case where (Y, ξ) is a boundary of some neighborhood of symplectic spheres plumbed together (Eliashberg,McDuff,Lisca,Ohta and Ono, Schöenberger,Starkson), or when they are supported by relatively simple planar open books (Plamenevskaya and Van-Horn Morris, Sivek and Van-Horn Morris, Kaloti and Li).

Theorem (Stipsicz)

If (Y, ξ) is symplectic cobordant to (S^3, ξ_{std}), then there are only a finite number of possible values of $\chi(X)$ and $\sigma(X)$.

Theorem (Etnyre)

If (Y, ξ) is supported by a planar open book then it is symplectic cobordant to (S^3, ξ_{std}).

Legendrian Surgery

Definition

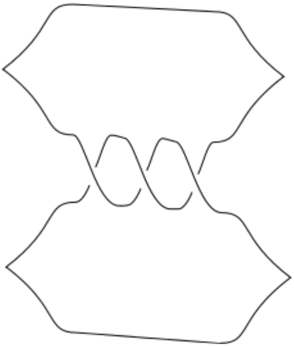
A Legendrian link L in (S^3, ξ_{std}) is an oriented link in S^3 with $TL \subset \xi_{std}$.

Theorem (Weinstein,Eliashberg)

Given any Legendrian link in (S^3, ξ_{std}) there is a natural way to associate a contact 3-manifold Legendrian surgery on L which is topologically some integral surgery on L and Stein fillable by the trace of the surgery.

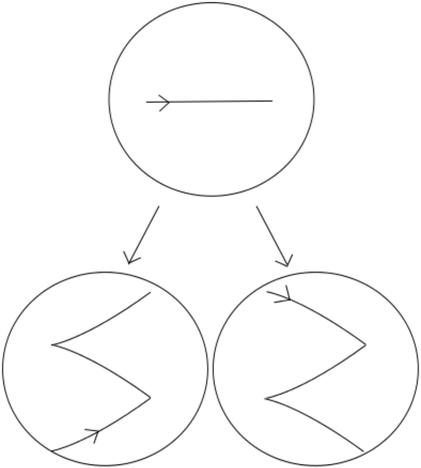
Front Diagrams

Legendrian links in S^3 have diagrams called front diagrams (invented by Arnold) which essentially determine the links up to isotopy through Legendrians.



Stabilization of a Legendrian Knot

The following two operations on Legendrian links (given from their front diagrams) are called (respectively positive and negative) stabilizations.



Theorem (Onaran)

If L is a Legendrian link in (S^3, ξ_{std}), then after a sufficient number of positive and negative stabilizations (s_+ and s_-) on L, $s_+^{n_1} s_-^{n_2}(L)$ can be embedded in the page of a planar open book which supports the standard contact structure on S^3. In particular Legendrian surgery on $s_+^{n_1} s_-^{n_2}(L)$ is supported by a planar open book.

Theorem (M)

If L is a Legendrian link with n components in (S^3, ξ_{std}), then after a sufficient number of positive and negative stabilizations (s_+ and s_-) on L, any Stein filling (X, ω) of Legendrian surgery on $s_+^{n_1} s_-^{n_2}(L)$ has $\chi(X) = 1 + n$ and $\sigma(X) = -n$.

An Open Question

Question

If L is a Legendrian link in (S^3, ξ_{std})*, then after a sufficient number of positive and negative stabilizations* (s_+ *and* s_-) *on L, is any Stein filling* (X, ω) *of Legendrian surgery on* $s_+^{n_1} s_-^{n_2}(L)$ *diffeomorphic to the trace?*

Thanks!

Thanks for listening.

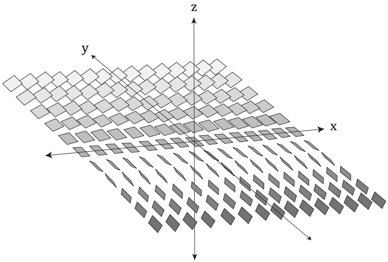
Algebraic Structures for Legendrian and Lagrangian Submanifolds with Generating Families

Ziva Myer

Bryn Mawr College

Tech Topology Conference 2016

Contact Manifold $(J^1M = T^*M \times \mathbb{R}, \xi)$



The standard contact structure on $\mathbb{R}^3$: $\xi = \ker(dz - ydx)$.

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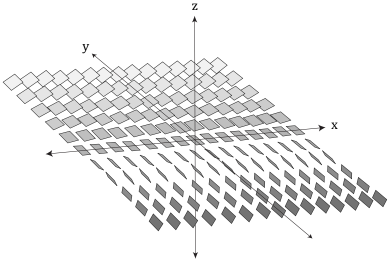

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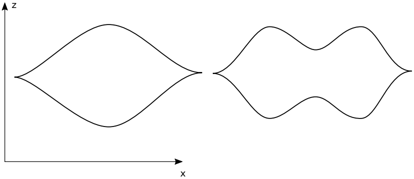
Legendrian submanifold $\Lambda \subset J^1M$
$T\Lambda \subset \xi = \ker(dz - \sum y_i dx_i)$.



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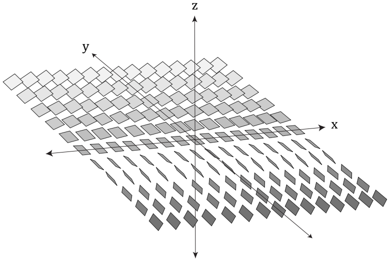

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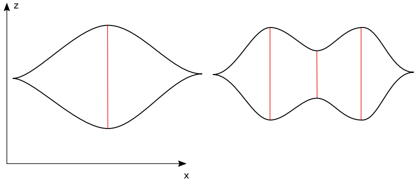

Important feature: Reeb Chords

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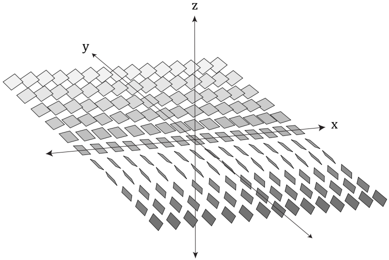

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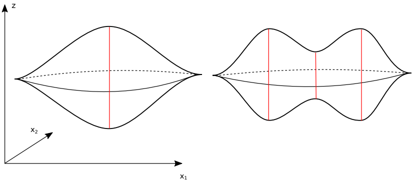

Important feature: Reeb Chords

Goal: Define algebraic invariants for Legendrians from Reeb chords.

Generating Family Cohomology

$$\Lambda \overset{F}{\rightsquigarrow} \{GH^*(F)\}_F$$

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Can we build additional invariant algebraic structure off of $GH^*(F)$ such as

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Motivating Questions

Can we build additional invariant algebraic structure off of $GH^*(F)$ such as

- ring structure?
- A_∞ algebra?
- A_∞ category?

Theorem (M.)

There exists a product on Generating Family Cohomology

$$\mu_2 : GH^i(F) \otimes GH^j(F) \to GH^{i+j}(F).$$

that is invariant under Legendrian isotopy:

$$\begin{array}{ccc} GH^i(F) \otimes GH^j(F) & \xrightarrow{\mu_2} & GH^{i+j}(F) \\ \downarrow \cong & & \downarrow \cong \\ GH^i(\widehat{F}) \otimes GH^j(\widehat{F}) & \xrightarrow{\widehat{\mu_2}} & GH^{i+j}(\widehat{F}) \end{array}$$

Theorem (in progress)

There exists maps

$$m_k : C^{i_1}(F) \otimes \cdots \otimes C^{i_k}(F) \longrightarrow C^{\sum_\ell i_\ell + k - 2}(F)$$

such that $(C(F), \{m_k\}_{k=1}^{\infty})$ *is an* A_∞ *algebra, i.e.,*

$$\sum_{i+j+\ell=k} m_{i+1+\ell} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes \ell}) = 0.$$

Furthermore, this A_∞ *algebra is invariant up to* A_∞ *quasi-isomorphism under Legendrian isotopy.*

A_∞ Structure from Generating Families

Technique: Morse Flow Trees

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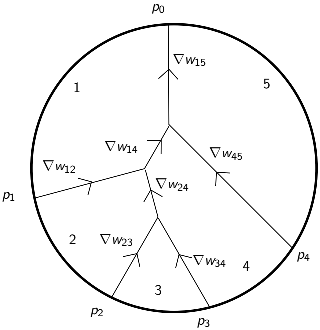
$m_k : C_+^{\otimes k}(w_F) \longrightarrow C_+(w_F)$ counts isolated trees:



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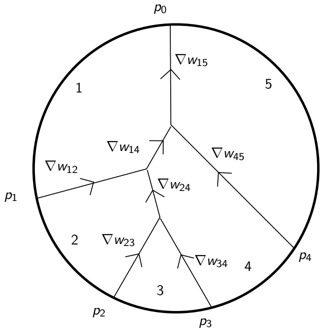

A_∞ relations come from compactifying 1-dimensional spaces of trees:

$$\sum_{i+j+k=l} m_{i+1+k} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes k}) = 0.$$

Future Work

Goal: Define A_∞ categories

- Objects: Generating families F
  - for Legendrians $\Lambda \subset J^1(M)$
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Thank you!

The Weinstein Conjecture for Iterated Planar Contact Structures

Bahar Acu

University of Southern California,
University of California, Los Angeles

Lightning Talks Session I
Tech Topology Conference
December 10, 2016

Goal

To study fillings of certain $(2n+1)$-dimensional contact manifolds by pseudoholomorphic curves and, by using this result, prove the Weinstein conjecture for that class.

Motivation

Theorem (Wendl, 2008)

Let $(M^3, \xi = \ker \lambda)$ *be a planar contact manifold. Then there exists an almost complex structure* J *on the symplectization* $\mathbb{R} \times M^3$ *such that* $(\mathbb{R} \times M^3, (e^s \lambda))$ *is foliated by embedded, finite energy, planar* J*-holomorphic curves of index 2.*

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Let $(M^3, \xi = \ker \lambda)$ be a planar contact manifold. Then there exists an almost complex structure J on the symplectization $\mathbb{R} \times M^3$ such that $(\mathbb{R} \times M^3, (e^s \lambda))$ is foliated by embedded, finite energy, planar J-holomorphic curves of index 2.

This result can be used in various applications to planar contact manifolds such as

- the Weinstein conjecture,
- equivalence and strong and Stein fillability.

Generalization attempt

Question

Can we do the same thing in higher dimensions?

Generalization attempt

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Answer

Not easy!

Automatic transversality and intersection theory do not exist in dim > 4.

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Remedy

Iterated planar Lefschetz fibrations.

Idea: carry 4-dimensional phenomena used to prove Wendl's theorem to higher dimensions inductively!

The fruit of the attempt

Theorem (A.)

Let (M^{2n+1}, ξ) be an iterated planar contact manifold. Then there exists a compatible J on $\mathbb{R} \times M$ such that $\mathbb{R} \times M$ is filled by planar finite energy J-holomorphic curves, i.e. **there exists a planar J-holomorphic curve through every point in $\mathbb{R} \times M$.**

The Weinstein Conjecture

Conjecture (Weinstein, 1978)

Every contact form on a closed $(2n+1)$-dimensional manifold has a closed Reeb orbit.

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Every contact form on a closed $(2n+1)$-dimensional manifold has a closed Reeb orbit.

It is **TRUE** when

- $\dim M{=}3$, ξ is overtwisted. (Hofer)
- $\dim M{=}3$, $\pi_2(M) \neq 0$, ξ is tight (Hofer)
- M is a solid torus (Etnyre, Ghrist)
- $\dim M{=}3$, ξ is supported by a planar open book. (Abbas, Cieliebak, Hofer)
- $\dim M{=}3$, λ is arbitrary. (Taubes)
- $\dim M{=}2n+1$, ξ is plastikstufe-overtwisted. (Albers-Hofer)

Iterated planar Lefschetz fibrations

Definition

A Weinstein domain (W^{2n}, ω), $n \geq 2$, admits an **iterated planar Lefschetz fibration** if

- there exists a sequence of Lefschetz fibrations $f_2, \ldots, f_n$ where $f_i : W^{2i} \to \mathbb{D}$ for $i = 2, \ldots, n$.
- Each regular fiber of f_{i+1} is the total space of f_i, i.e., W^{2i} is a regular fiber of f_{i+1}.
- $f_2 : W^4 \to \mathbb{D}$ is a planar Lefschetz fibration.

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Examples

1) $W = T^*S^n$ *since* $T^*S^2 \subset T^*S^3 \subset \cdots \subset T^*S^n$.
2) A_k*-singularity:* $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1^2 + \ldots, z_{n-1}^2 + z_n^{k+1} = 1\} \subset (\mathbb{C}^n, \omega_{std})$

The Weinstein Conjecture in Higher Dimensions

An **iterated planar contact manifold** = a contact manifold supporting an open book whose pages admit an iterated planar Lefschetz fibration.

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Theorem (A.)

Let (M, ξ) *be a* $(2n+1)$*-dimensional iterated planar contact manifold. Then* M *satisfies the Weinstein conjecture.*

Thanks!

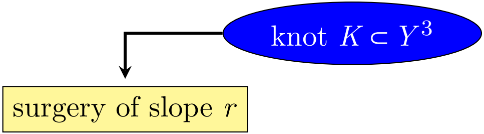
$PSL_2(\mathbb{C})$ Character variety and Dehn surgeries

Huygens C. Ravelomanana

University of Georgia

December 10, 2016

knot $K \subset Y^3$


knot $K \subset Y^3$
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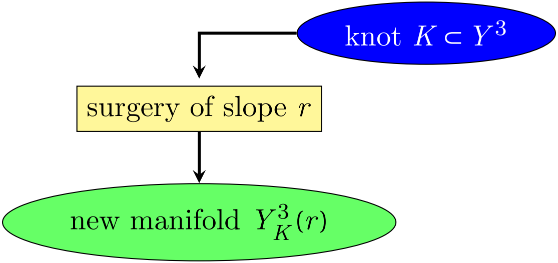
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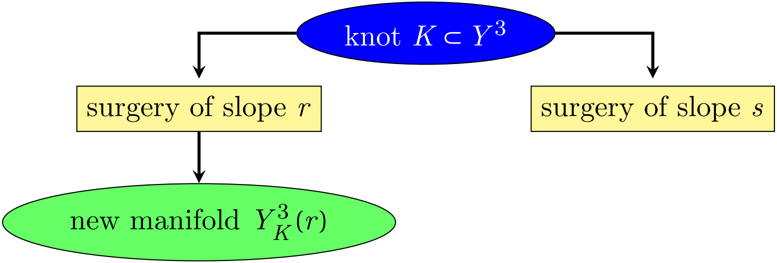
knot $K \subset Y^3$
surgery of slope r
surgery of slope s
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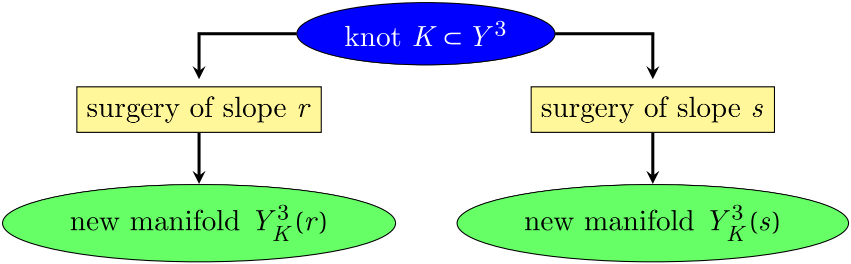
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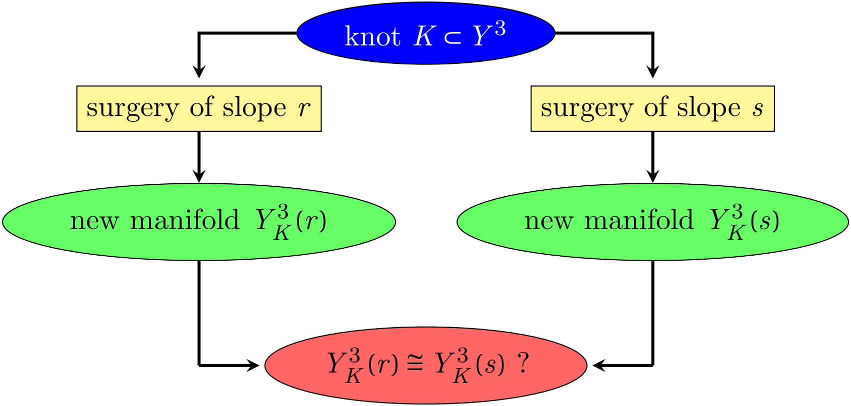
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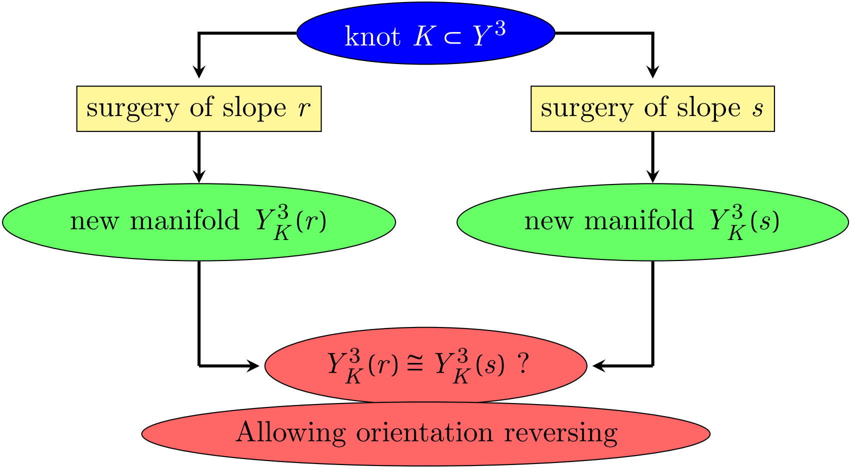
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Allowing orientation reversing

Examples

If K is the unknot in S^3, then $S^3_K(p/q) = L(p,q)$

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for relatively prime pairs of integers (p, q_1) and (p, q_2).

Assume $Y_K := Y \setminus int(\mathcal{N}(K))$ **is boundary irreducible and irreducible.** (This exclude the unknot in S^3 case.)

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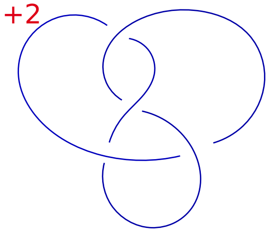

(Small Seifert-fibered)

Let's fix a slope s and define

$$C(s) = \left\{\text{slope} \quad r \neq s | \; Y_K(r) \cong Y_K(s)\right\}.$$

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Question: When is $C(s) \neq \emptyset$?

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Observation

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If $Y = S^3$ and K is an amphicheiral knot then $C(s) \neq \emptyset$.
Moreover $C(s) = \{-s\}$ for all known cases.

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Moreover $C(s) = \{-s\}$ for all known cases.

Main Question

Do we have $\sharp C(s) \leq 1$ in general ?

Main result

Main result

Theorem (R.)

Let's assume $Y_K(s)$ is small-Seifert. If $\mathrm{Hom}(\pi_1(Y), PSL_2(\mathbb{C}))$ contains only diagonalisable representations and $||s||_{CS}$ is not a multiple of $s \cdot \lambda$. Then $\sharp C(s) \leq 1$.

Main result

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Here , $||\ ||_{CS}$ is a semi-norm on $H_1(\partial Y_K;\mathbb{R})$ similar to the Culler-Shalen semi-norm and λ is the rational longitude of K.

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Here , $|| \; ||_{CS}$ is a semi-norm on $H_1(\partial Y_K; \mathbb{R})$ similar to the Culler-Shalen semi-norm and λ is the rational longitude of K.

The norm $||s||_{CS}$ is the degree count of a regular function

$$f_s : \widetilde{X}(Y_K) \to \mathbb{C}, \quad \chi \mapsto \chi(s)^2 - 4$$

restricted to one-dimensional components.

Proof

Proof

- If $r \in C(s)$ then $||r||_{CS} = ||s||_{CS}$ provided that $\text{Hom}(\pi_1(Y), PSL_2(\mathbb{C}))$ contains only diagonalisable representations.

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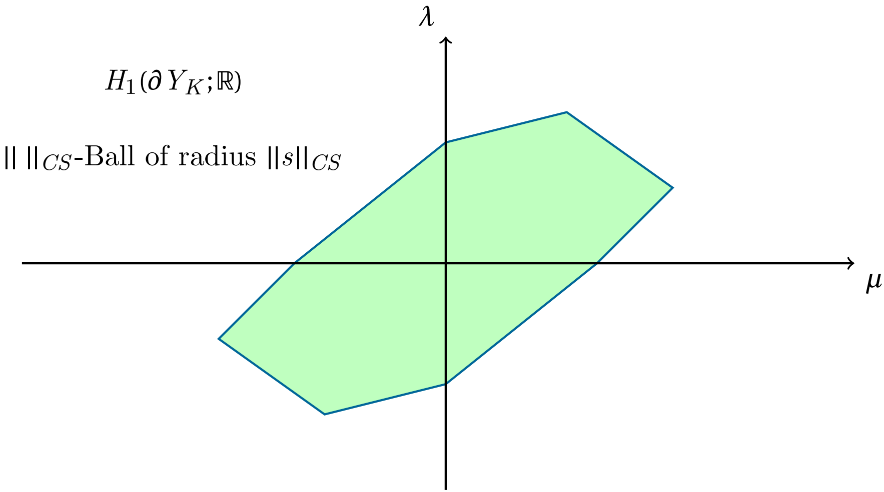
- If $||s||_{CS}$ is not a multiple of $s \cdot \lambda$ then the line determined by r and s passes through the interior of the $||\ \ ||_{CS}$-ball of radius $||s||_{CS}$.

Picture



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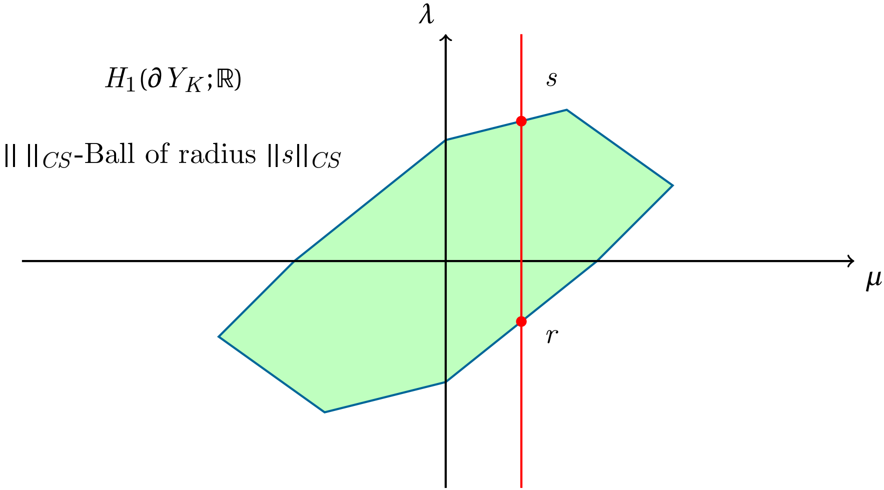

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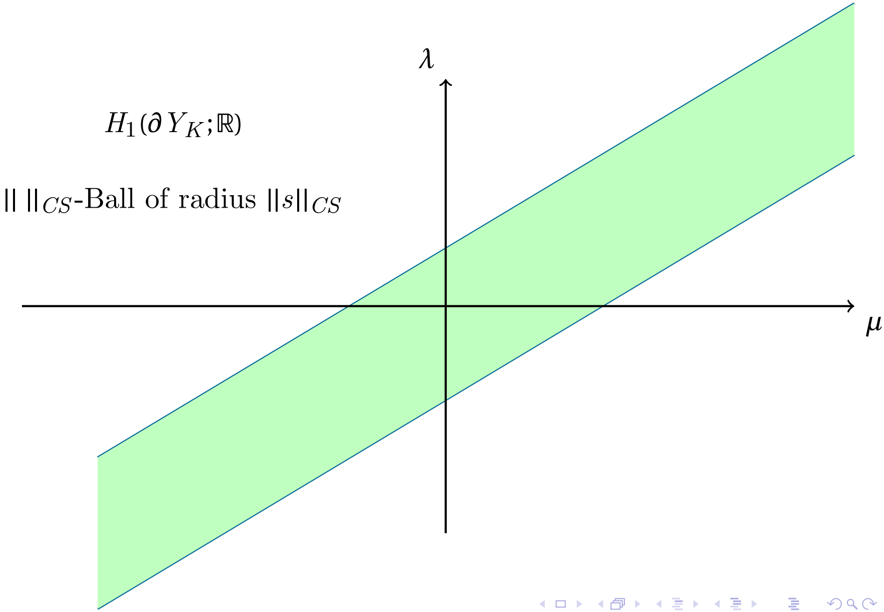

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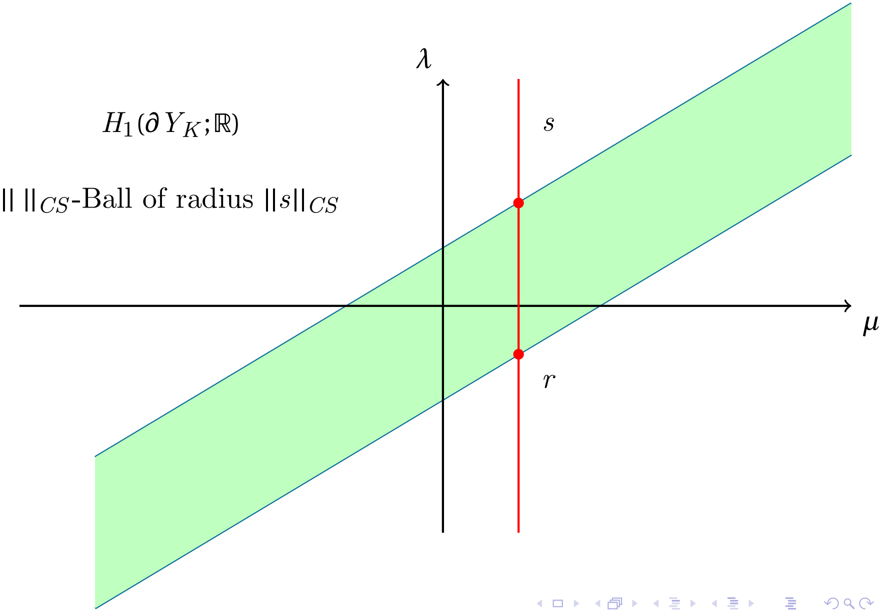
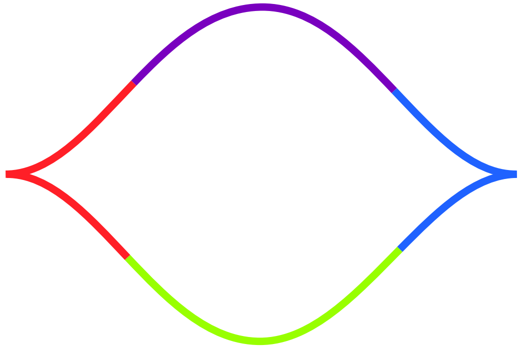


Thank You!

Immersed Lagrangian Fillings of Legendrian Knots

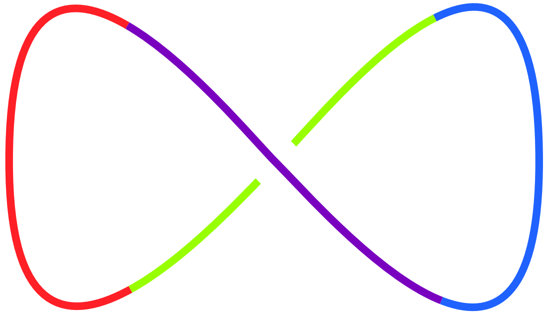


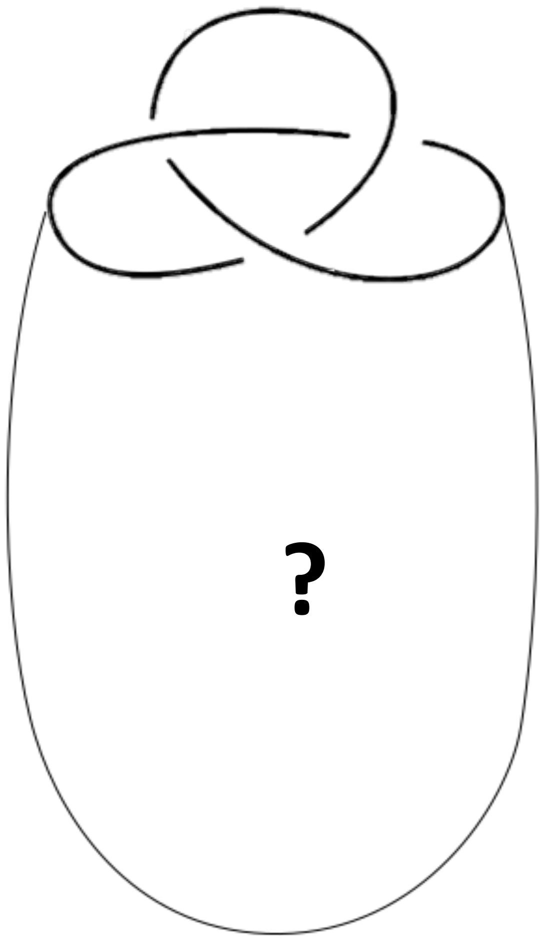
Tech Topology Conference

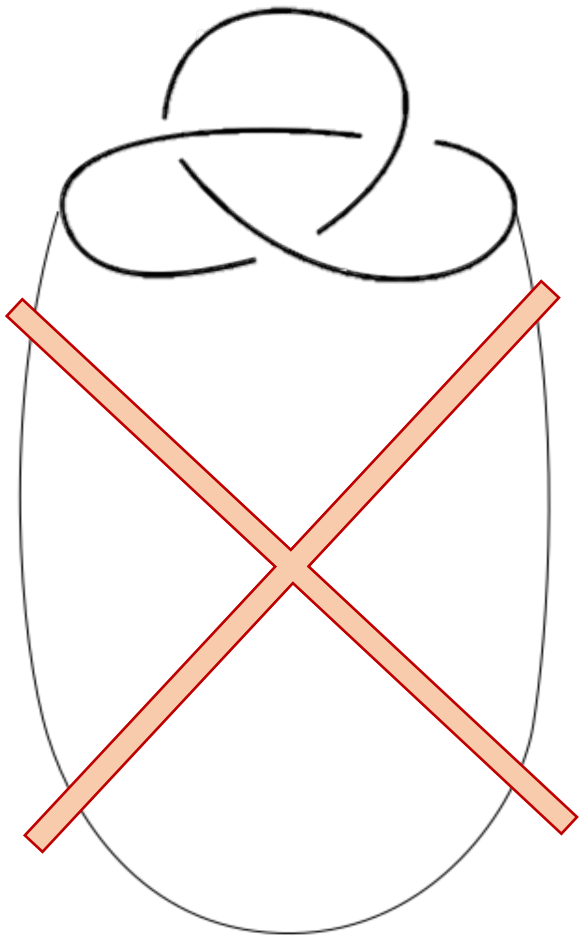
December 10, 2016

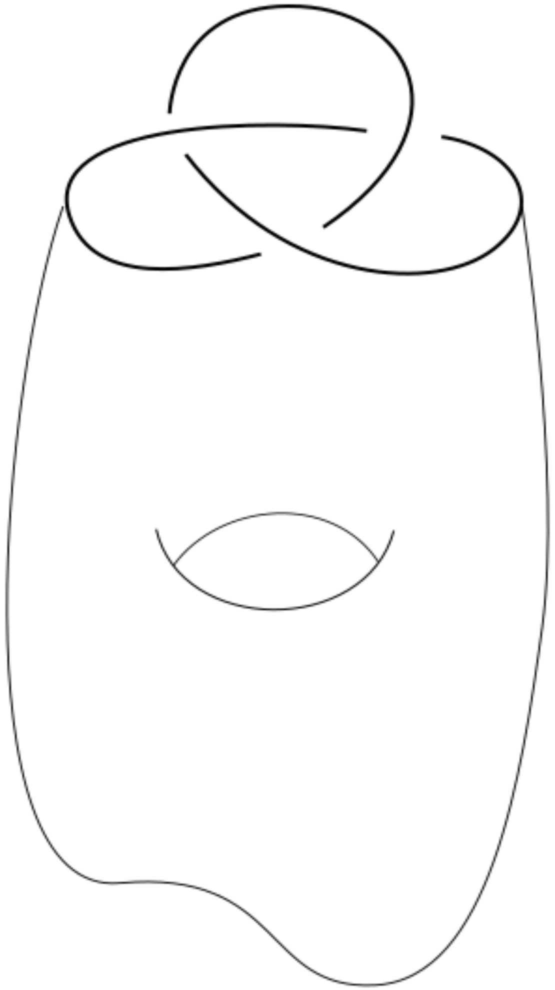
Samantha Pezzimenti

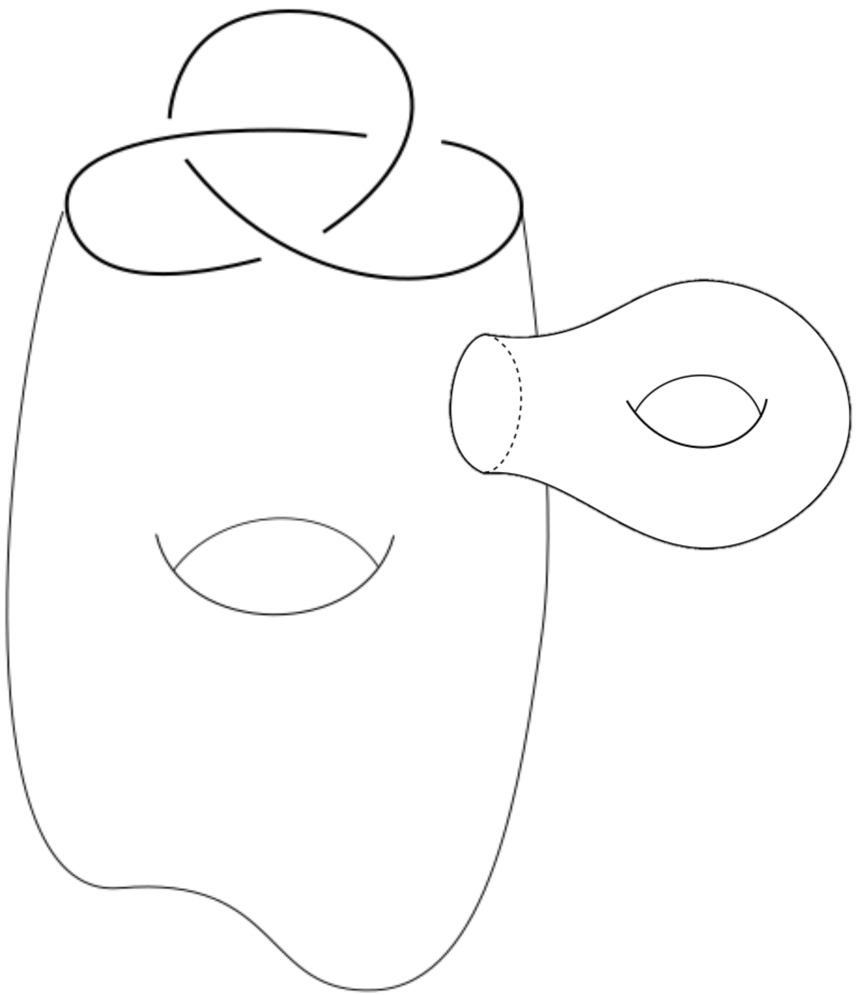
Bryn Mawr College

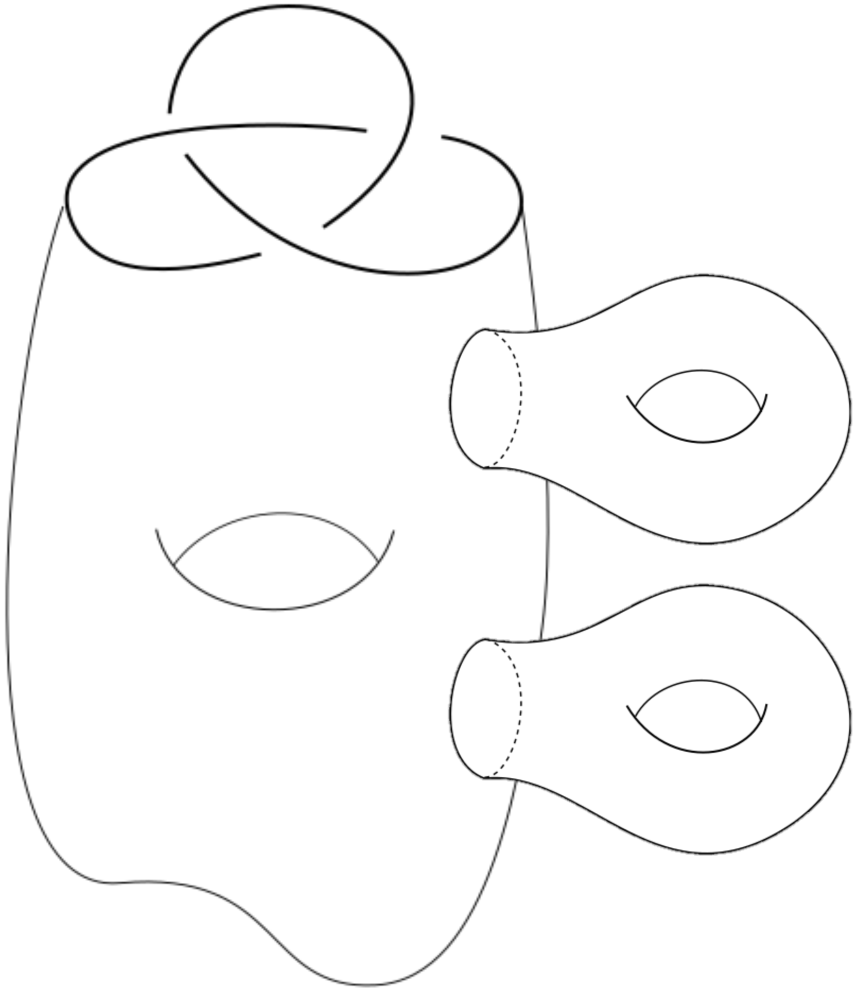


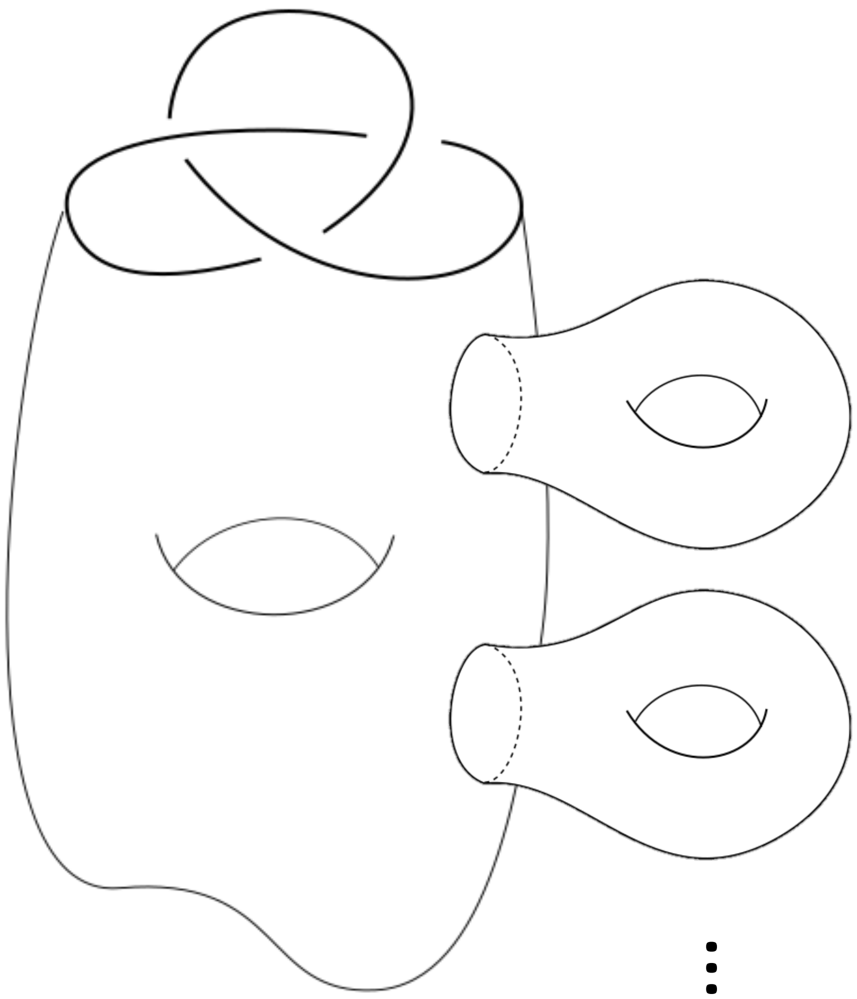


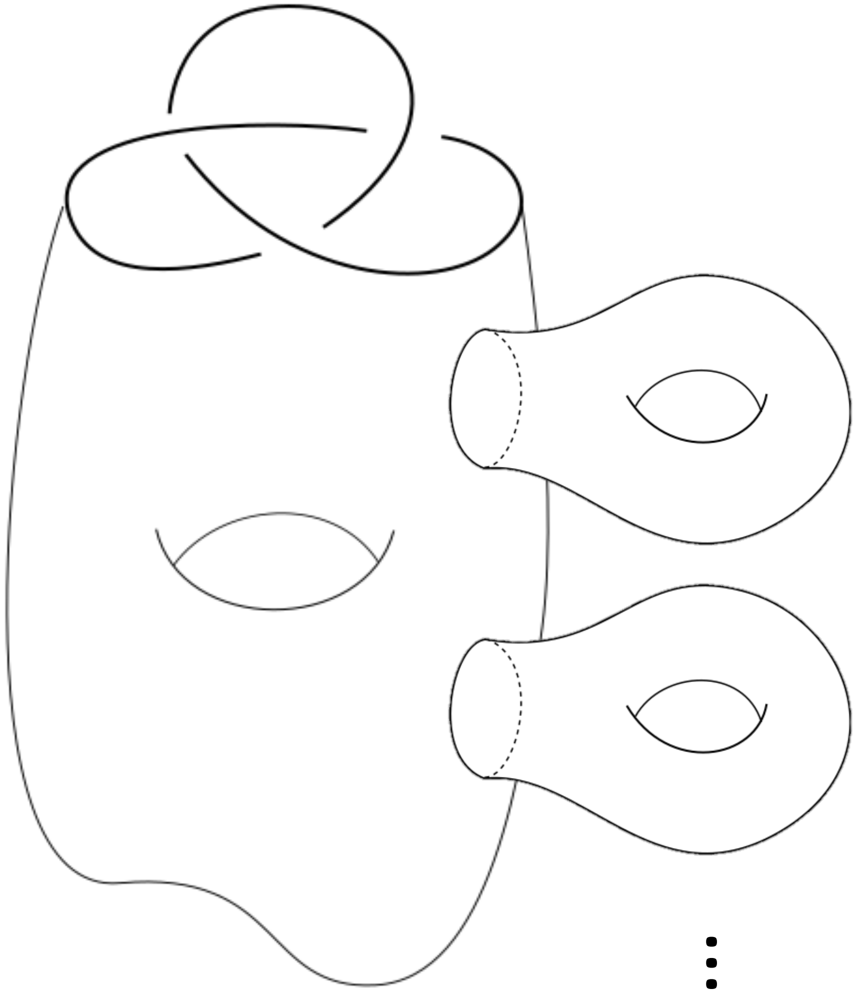




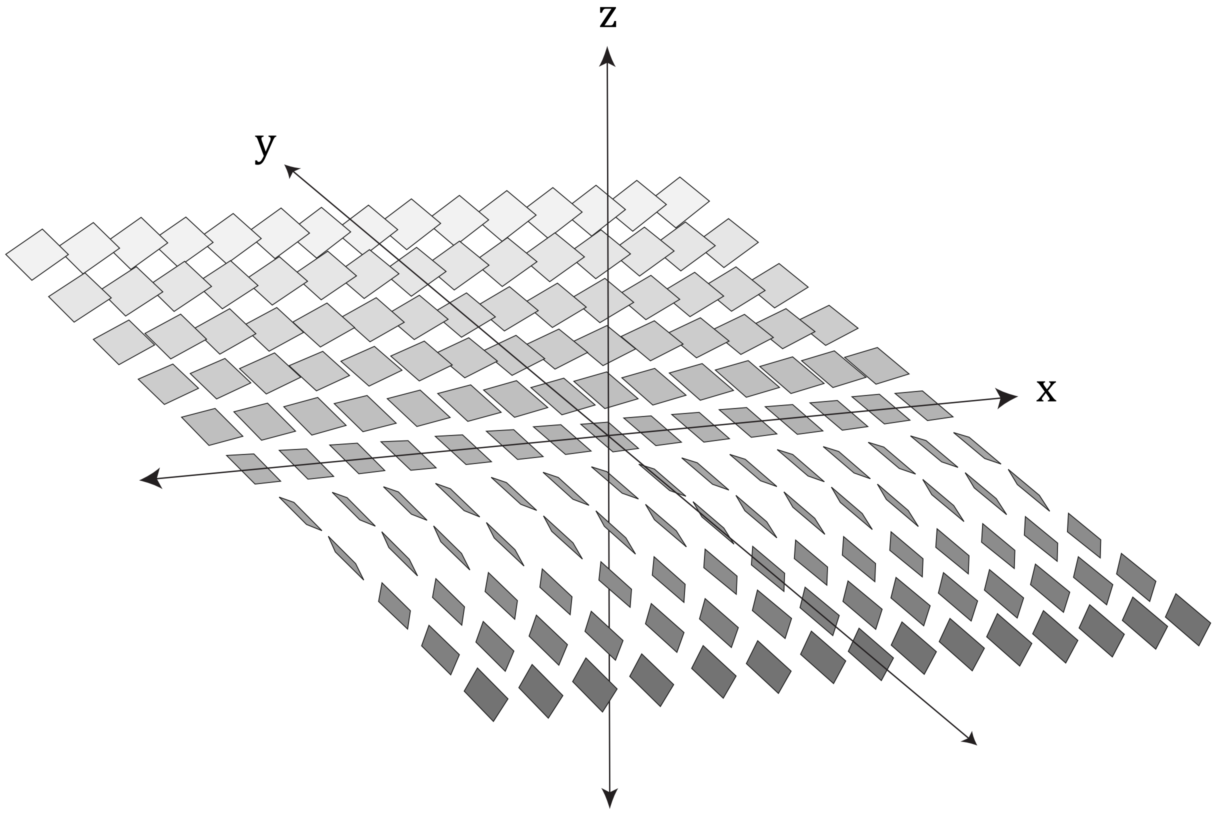






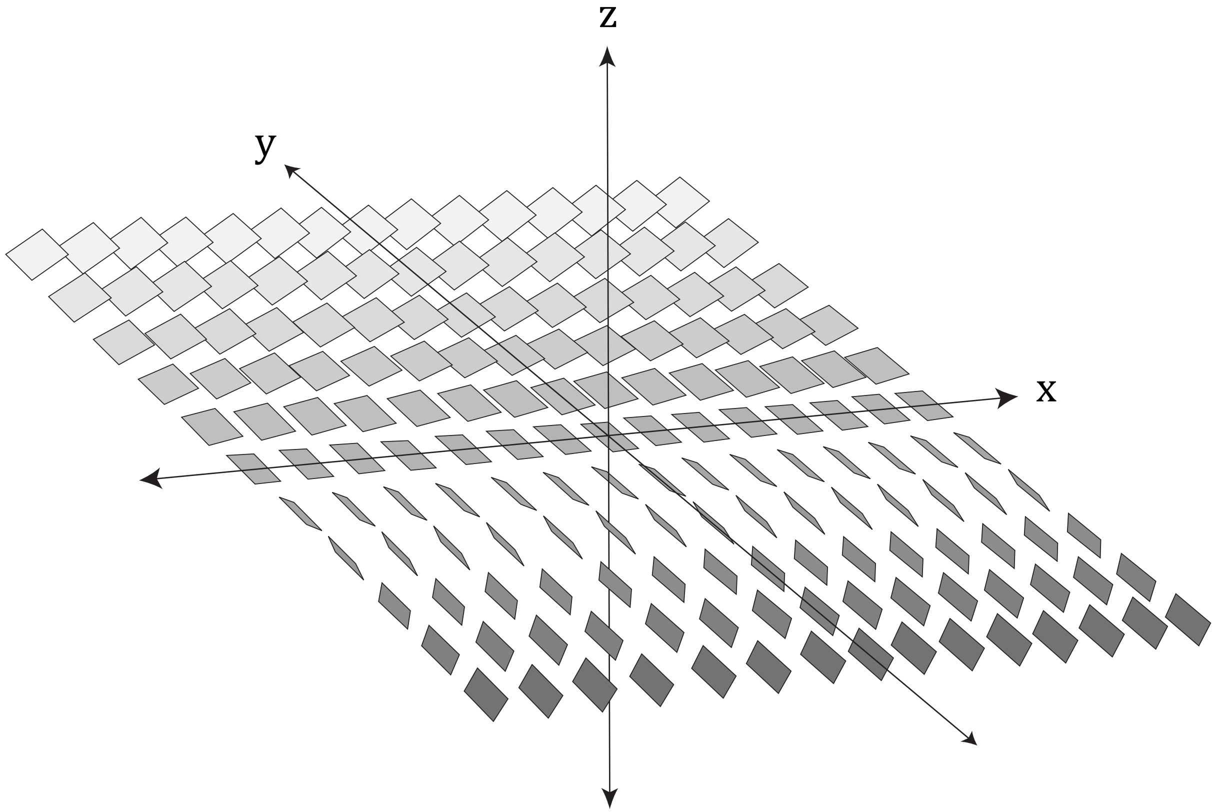


The genus of a smooth filling is not determined by the knot.



Contact Manifold:

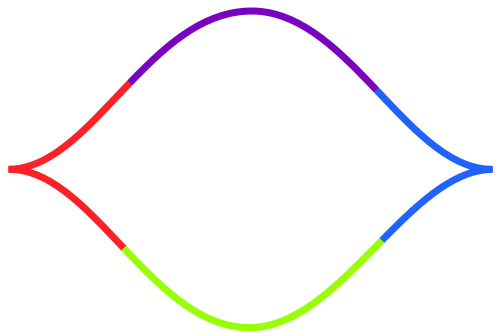
$(\mathbb{R}^3,\ \xi = ker(dz - ydx))$



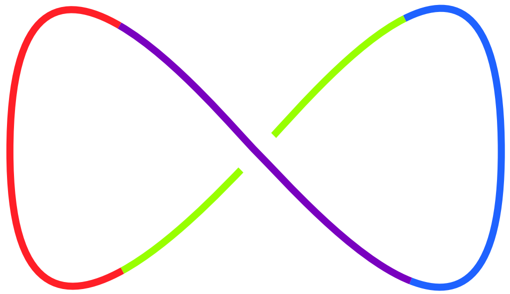
Contact Manifold:

$(\mathbb{R}^3,\ \xi = ker(dz - ydx))$

A knot is **Legendrian** if all of its tangent vectors lie in the planes of the contact structure.



Front Projection (xz)



Lagrangian Projection (xy)

Contact Manifold:

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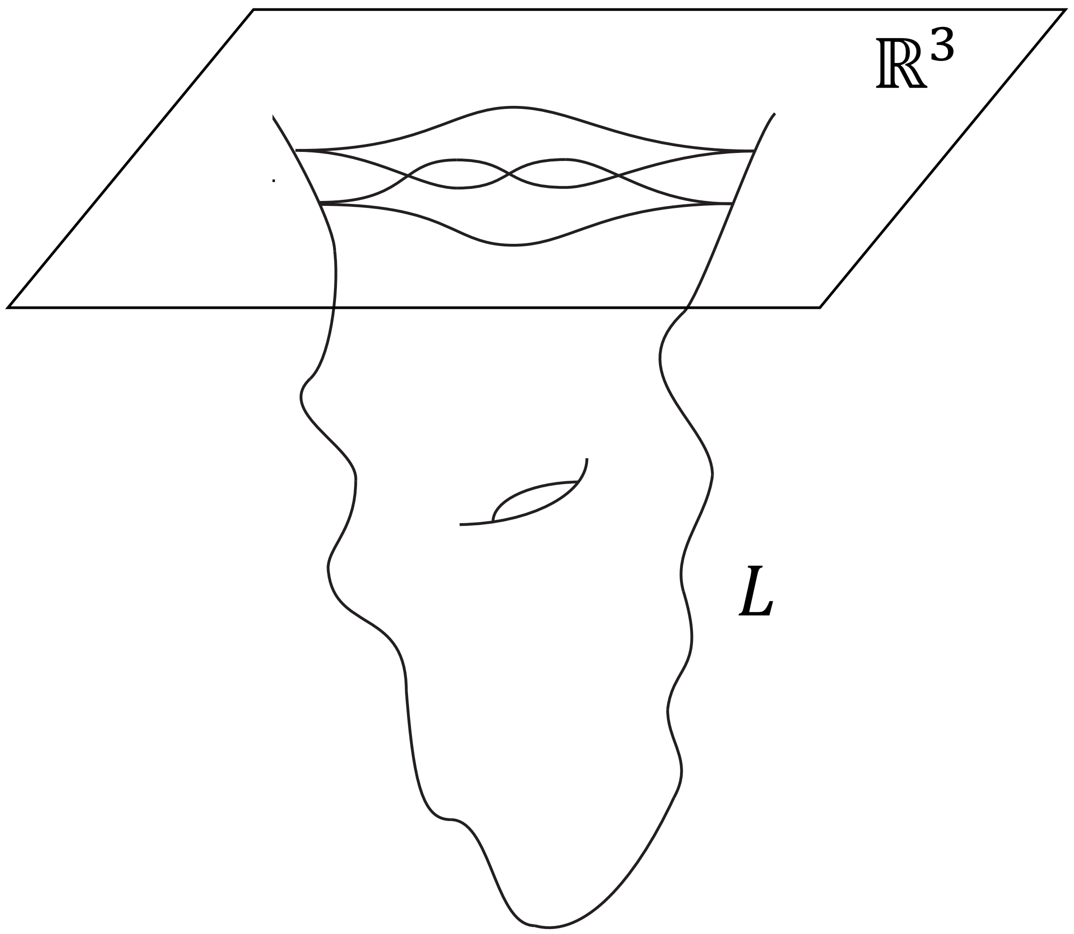
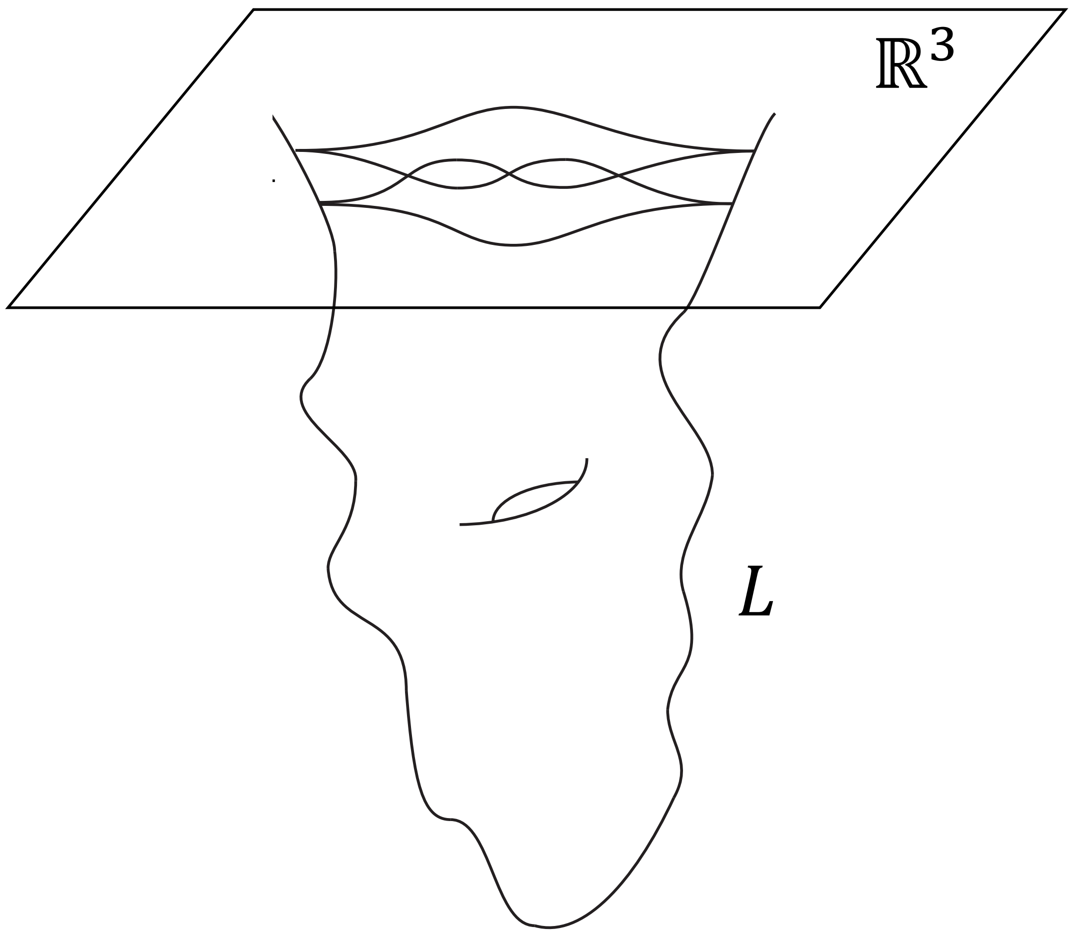


Symplectic Manifold:

$(\mathbb{R}\times\mathbb{R}^3, \omega = d(e^t\alpha))$



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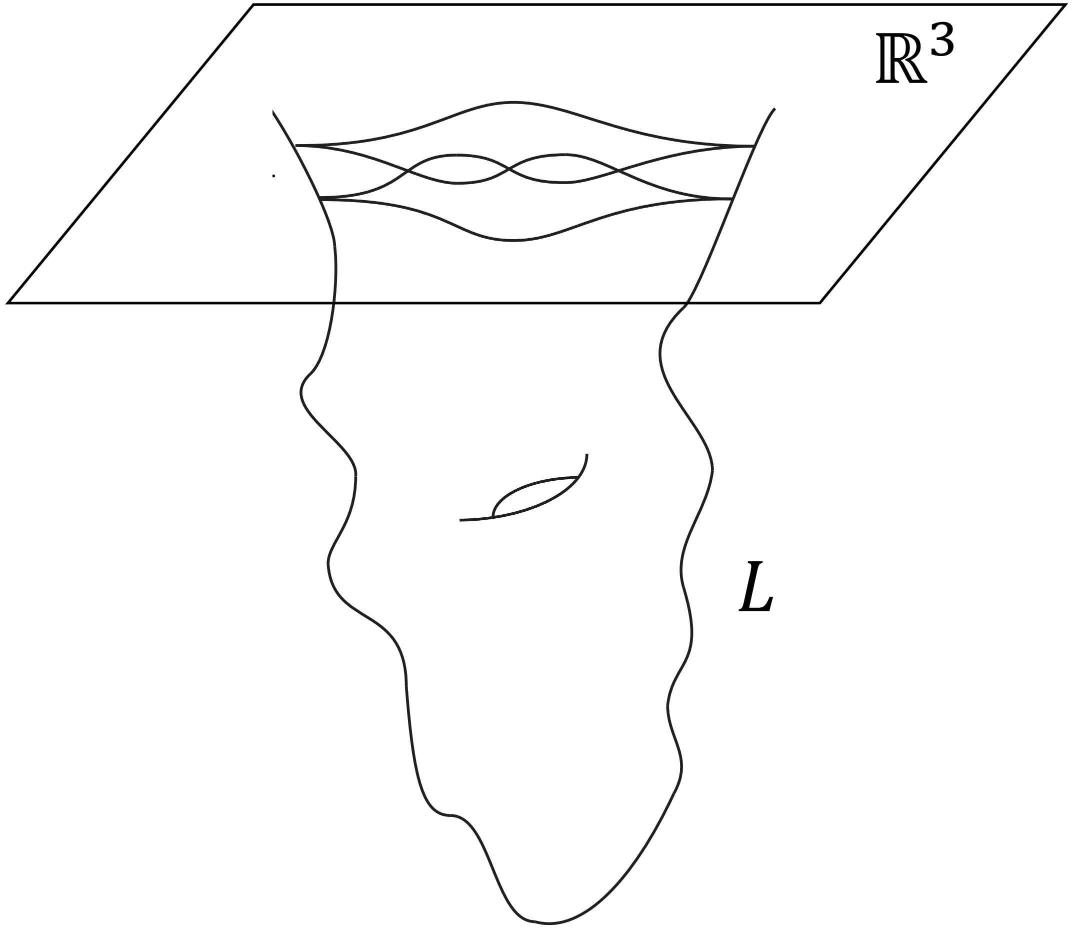
$\omega(\vec{v},\vec{w}) = 0, \forall \vec{v},\vec{w} \in T_pL$



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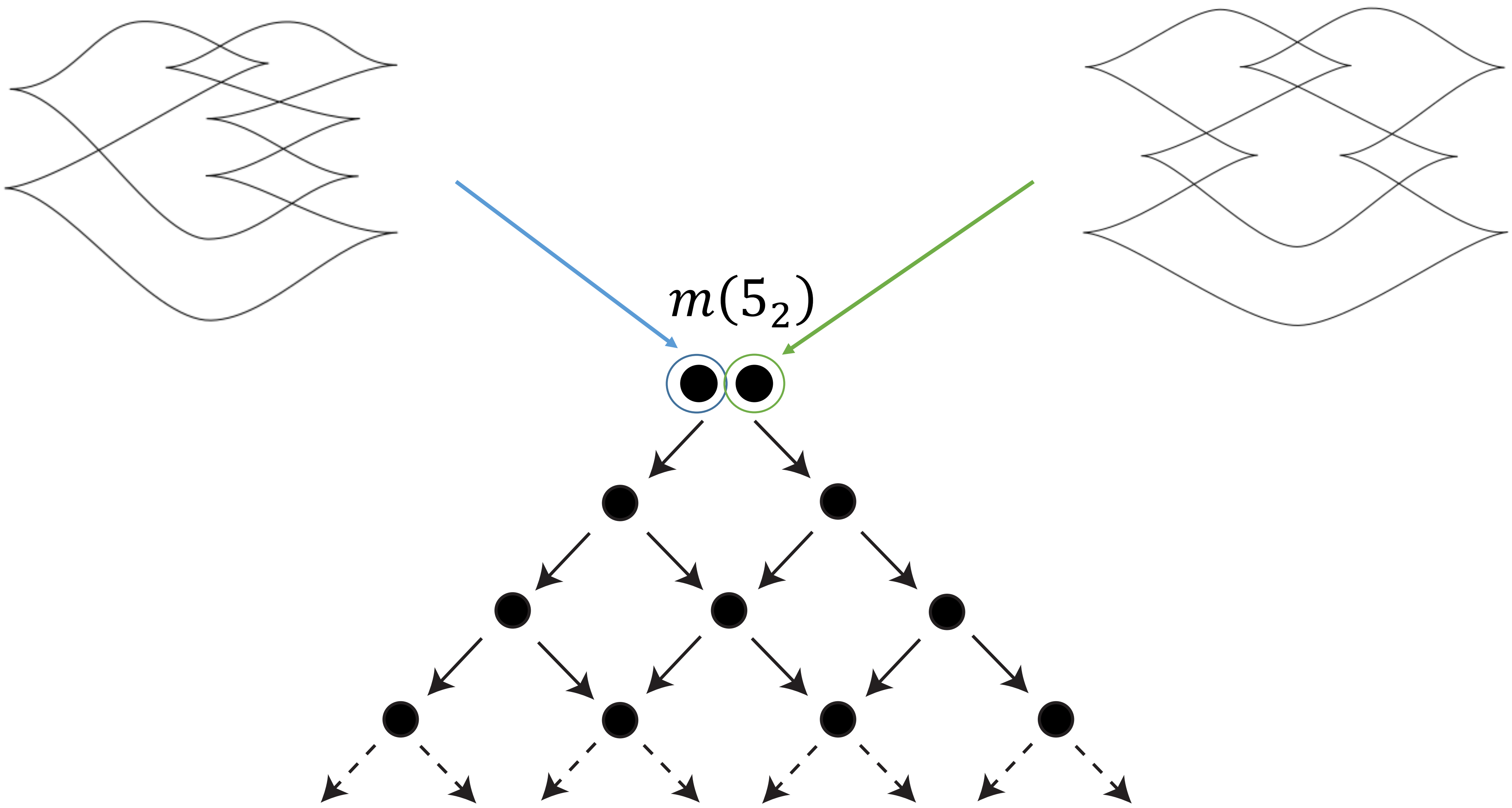
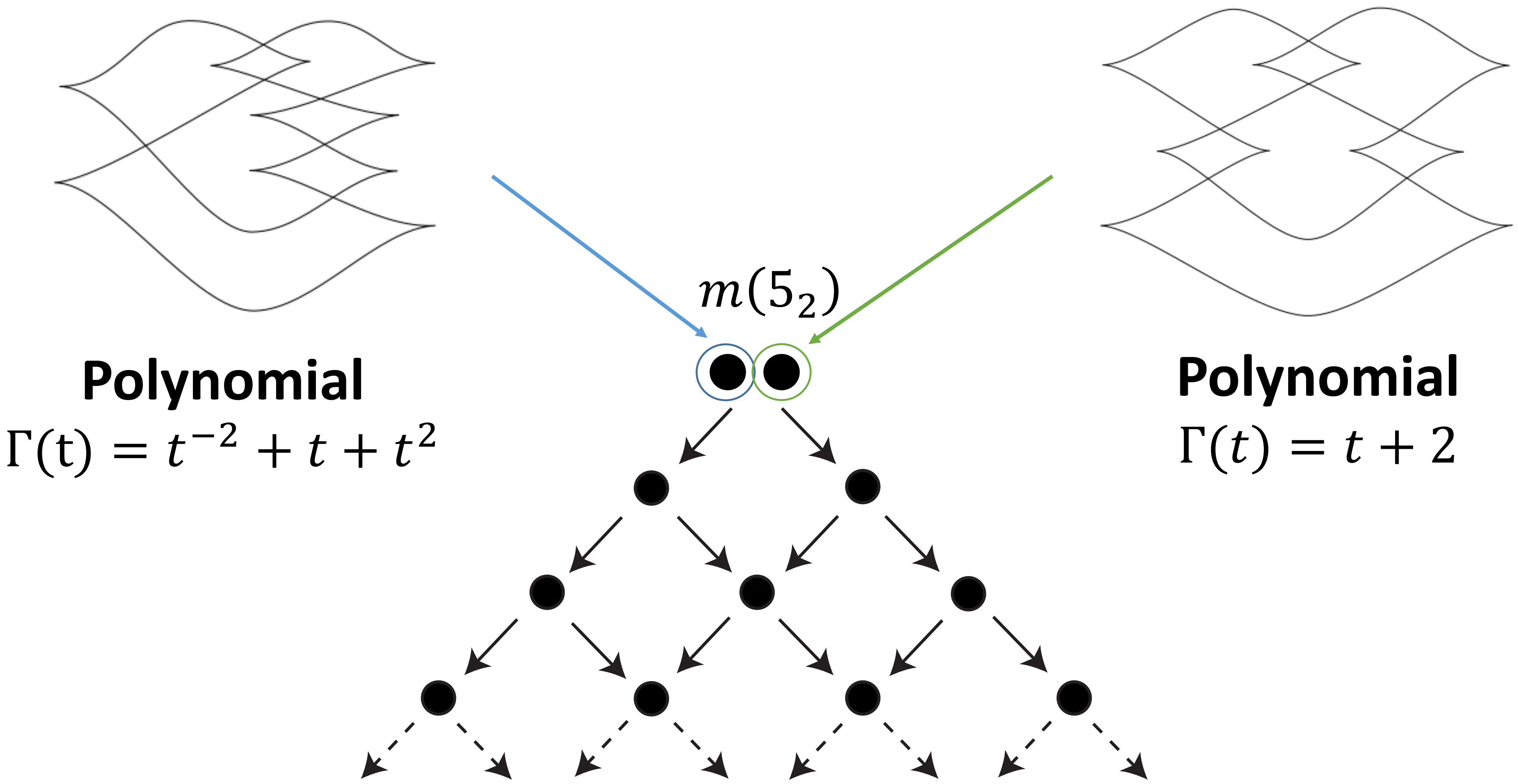
Our Lagrangian fillings:

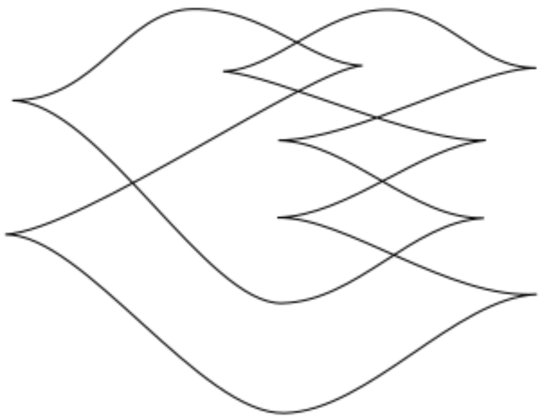
Exact, Maslov 0



Mountain Range for $m(5_2)$



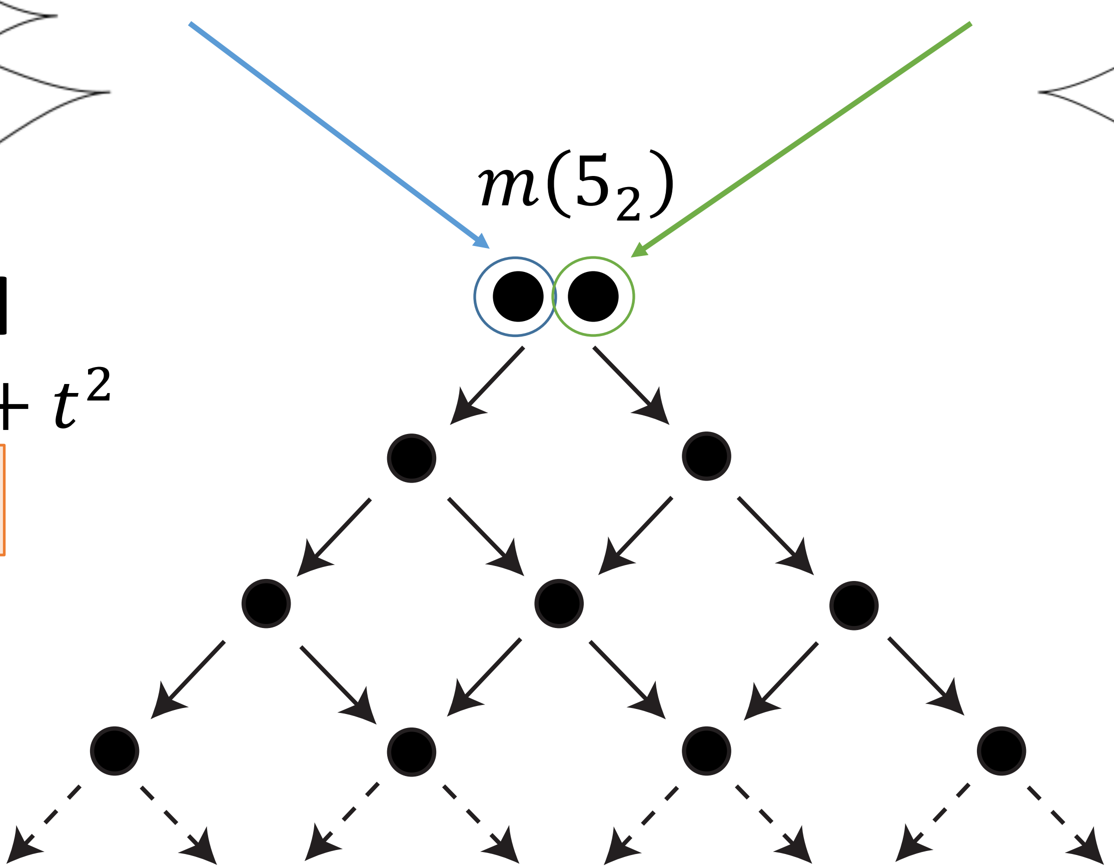
$m(5_2)$
Polynomial
$\Gamma(\mathrm{t}) = t^{-2} + t + t^2$
Polynomial
$\Gamma(t) = t + 2$

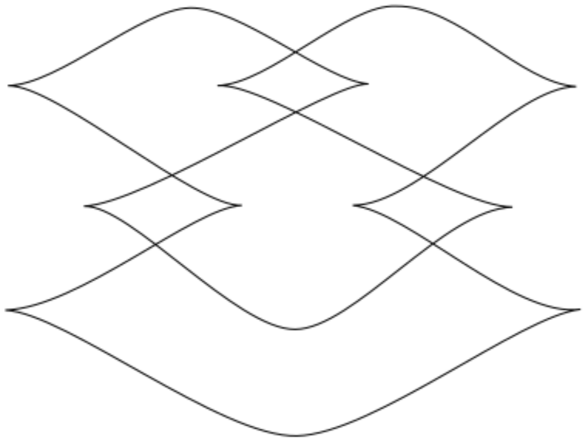


Polynomial

$\Gamma(\mathrm{t}) = t^{-2} + t + t^2$

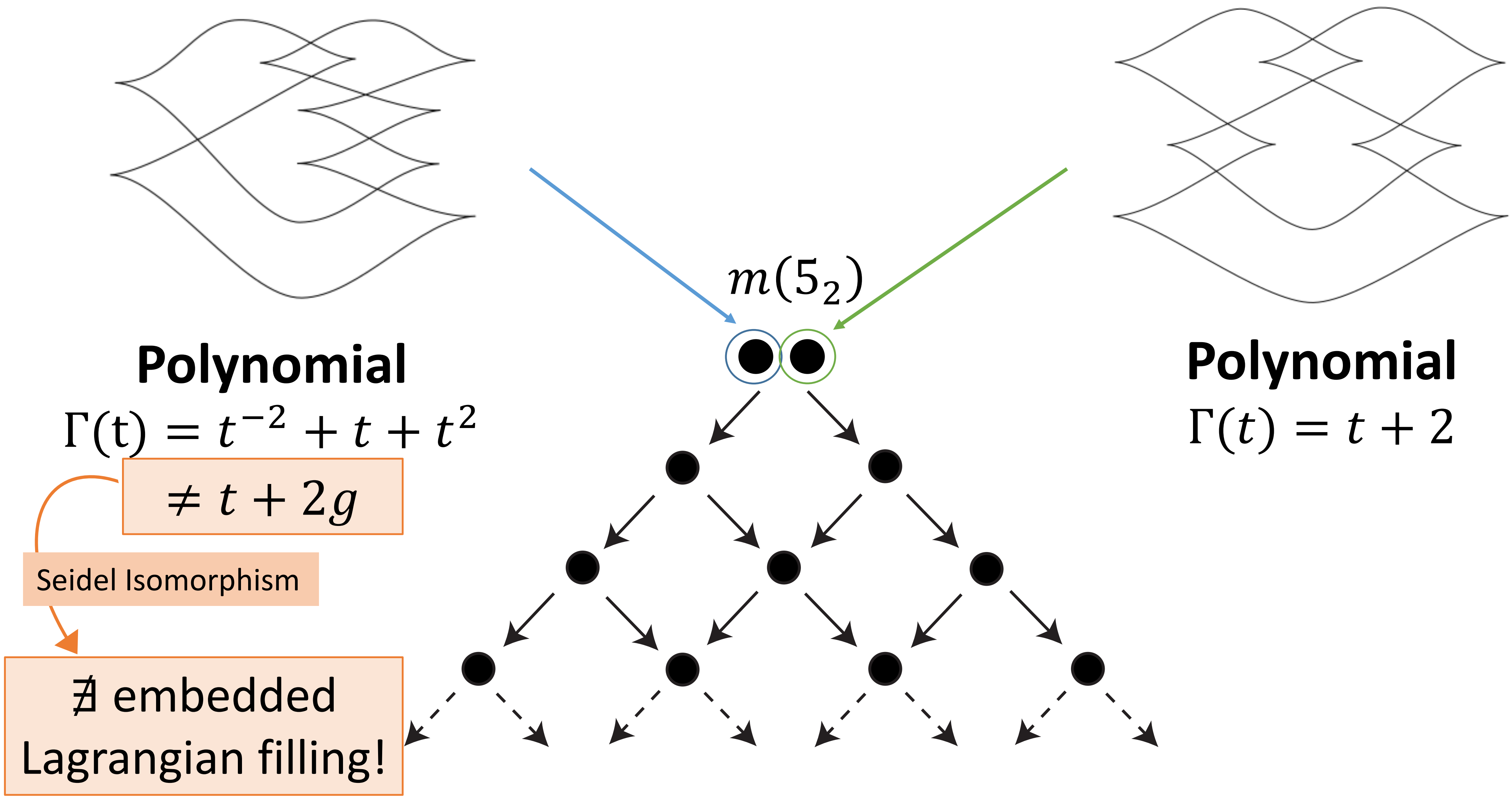
$\neq t + 2g$





Polynomial

$\Gamma(t) = t + 2$



$m(5_2)$
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$\neq t + 2g$
Seidel Isomorphism
∄ embedded
Lagrangian filling!
Polynomial
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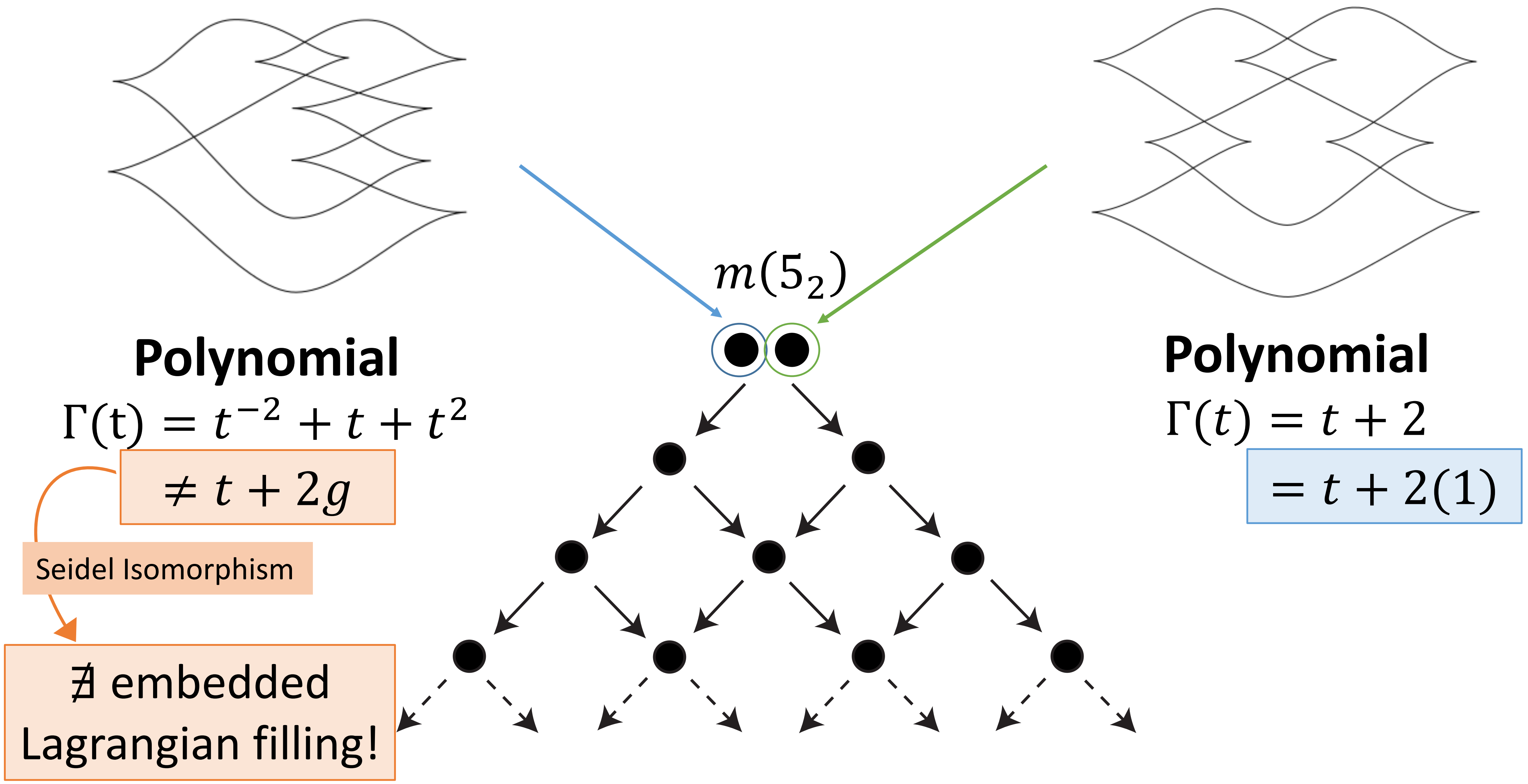

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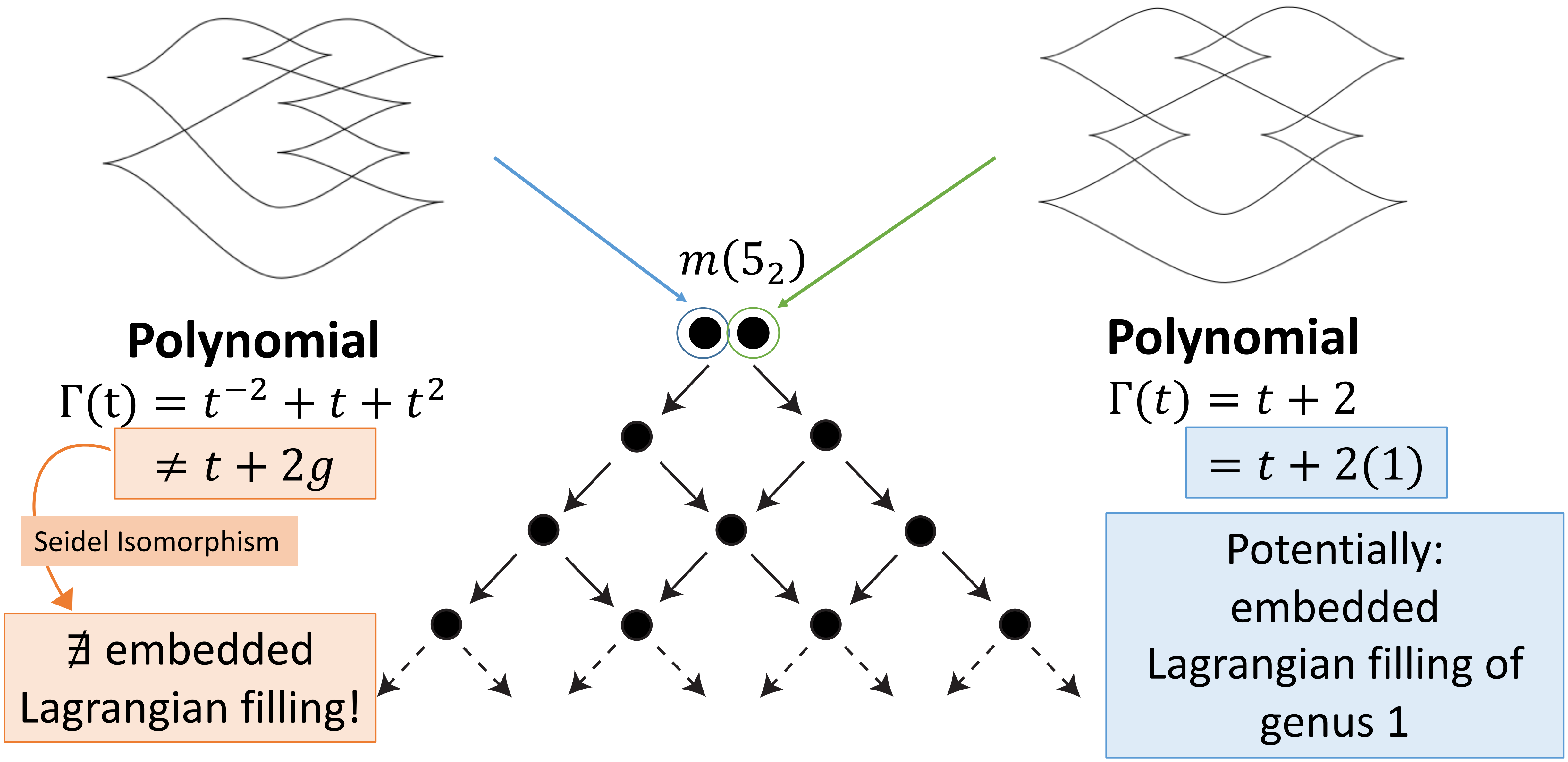
$\Gamma(t) = t + 2$
$= t + 2(1)$



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$\nexists$ embedded Lagrangian filling!
Polynomial
$\Gamma(t) = t + 2$
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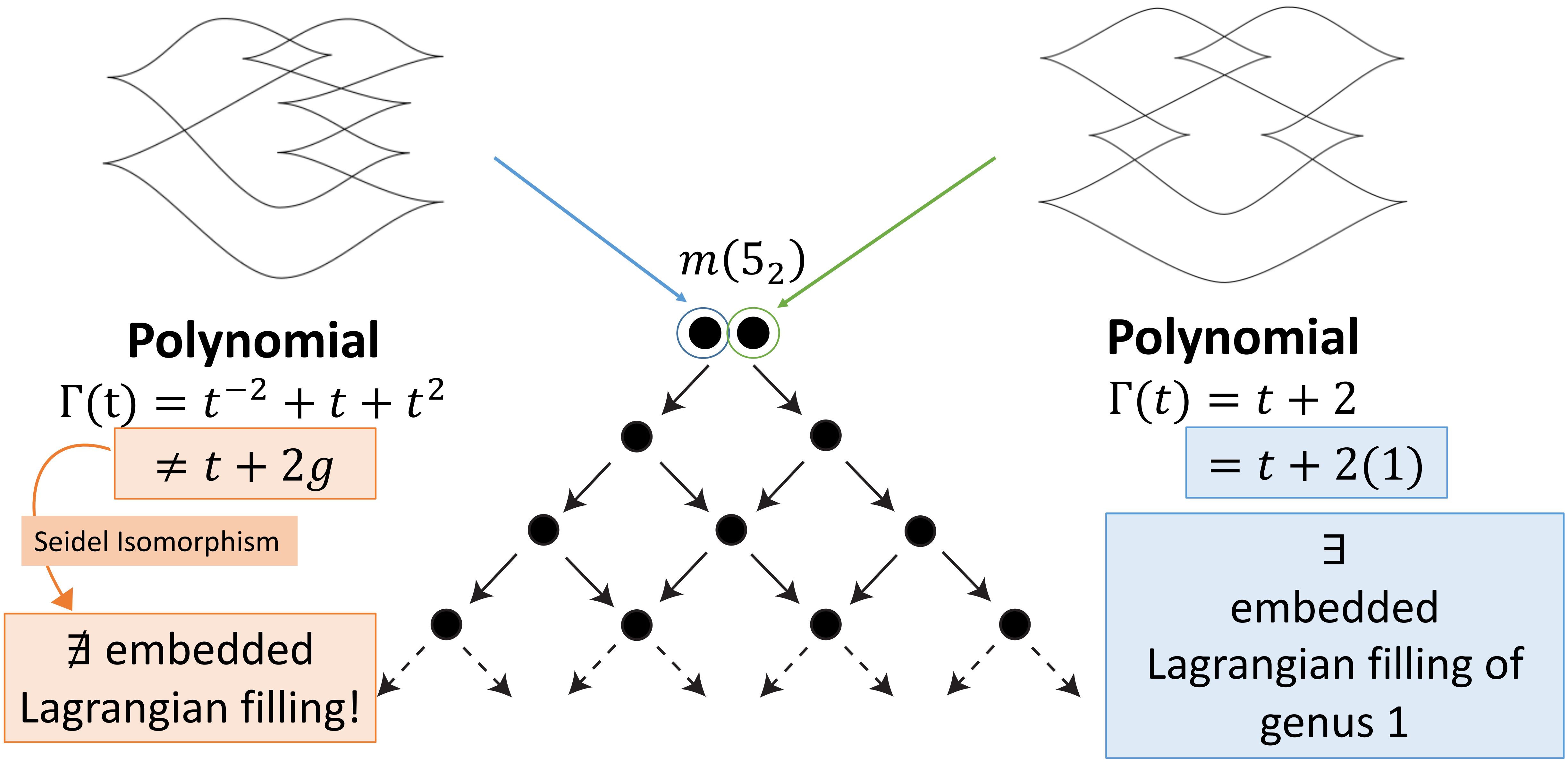
Potentially: embedded Lagrangian filling of genus 1



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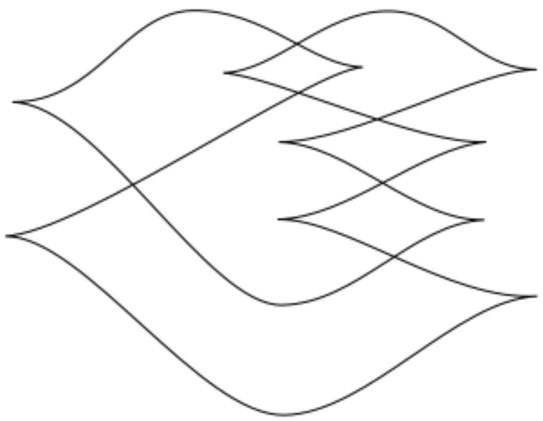
Seidel Isomorphism
∄ embedded Lagrangian filling!
Polynomial
$\Gamma(t) = t + 2$
$= t + 2(1)$
∃ embedded Lagrangian filling of genus 1

What does the polynomial of a Legendrian knot tell us about the genus/immersion points of an **immersed Lagrangian filling**?

My Research Results



Example: $\Gamma(\mathrm{t}) = 1t^{-2} + +0t^{0} + t + 0t^{-0} + 1t^{2}$

My Research Results

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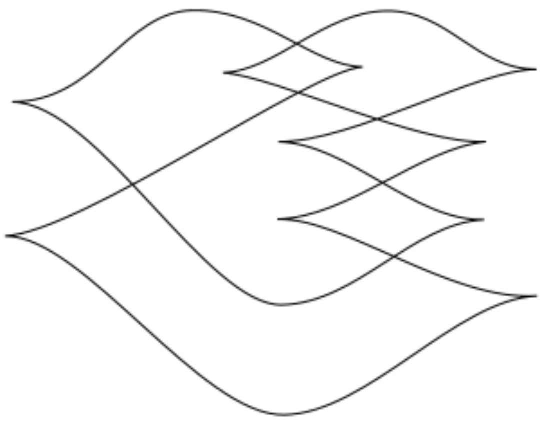
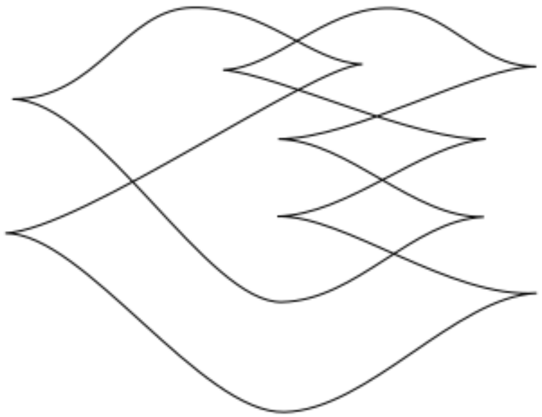


- Any immersed filling has at least 1 immersion point of index 2.

My Research Results

Example: $\Gamma(\mathrm{t}) = 1t^{-2} + +0t^{0} + t + 0t^{-0} + 1t^{2}$



- Any immersed filling has at least 1 immersion point of index 2.
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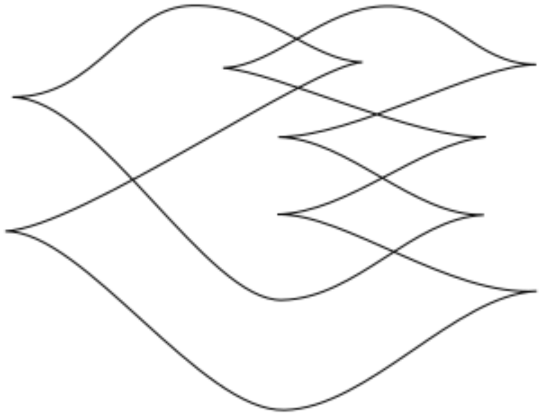
My Research Results



Example: $\Gamma(\mathrm{t}) = 1t^{-2} + +0t^{0} + t + 0t^{-0} + 1t^{2}$

- Any immersed filling has at least 1 immersion point of index 2.
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genus 1 filling with an additional
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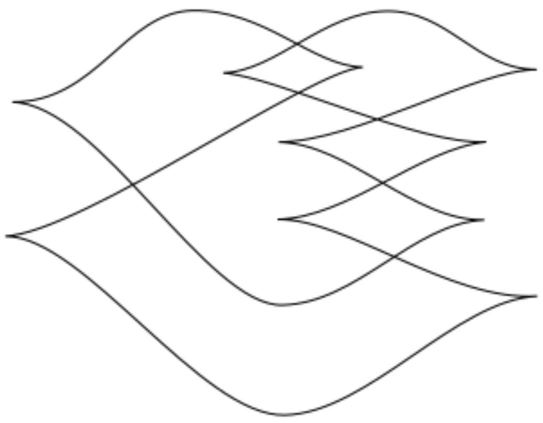
My Research Results

Example: $\Gamma(\mathrm{t}) = 1t^{-2} + +0t^{0} + t + 0t^{-0} + 1t^{2}$



- Any immersed filling has at least 1 immersion point of index 2.
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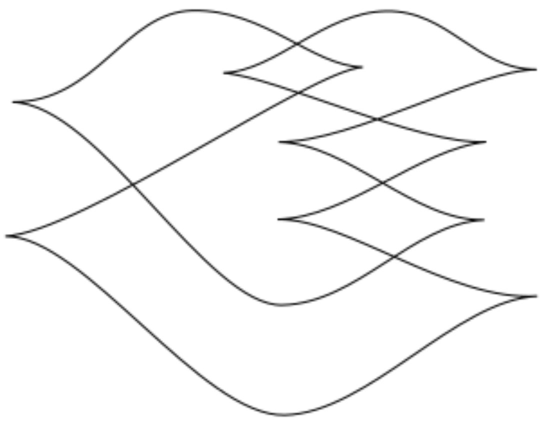
genus 1 filling with an additional immersion point of index 1

genus g filling with an additional g immersion points of index 1

My Research Results



Example: $\Gamma(\mathrm{t}) = 1t^{-2} + +0t^{0} + t + 0t^{-0} + 1t^{2}$

- Any immersed filling has at least 1 immersion point of index 2.
- Potentially immersed genus 0 filling

  genus 1 filling with an additional immersion point of index 1

  genus g filling with an additional g immersion points of index 1

- Can add more immersion points in pairs of consecutive indices.

Thank you!

Exact Lagrangian fillings of Legendrian $(2, n)$ torus links

Yu Pan

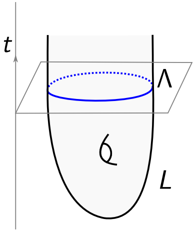
Duke University

Tech Topology Conference
Dec. 10, 2016

Exact Lagrangian fillings

For a Legendrian knot Λ in $(\mathbb{R}^3, \ker \alpha)$, where $\alpha = dz - ydx$, an exact Lagrangian filling of Λ is a 2-dimensional surface L in $(\mathbb{R}_t \times \mathbb{R}^3, \omega = d(e^t \alpha))$ such that

- L is cylindrical over Λ when t is big enough;
- there exists a function $f : L \to \mathbb{R}$ such that $e^t \alpha \Big|_{TL} = df$ and f is constant on Λ.



Questions

Given a Legendrian knot, we can ask the following questions.

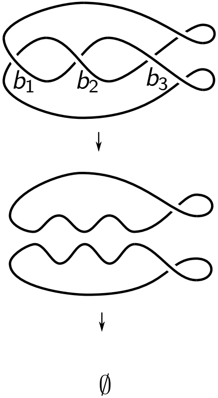
- Does it have an exact Lagrangian filling?
- What does an exact Lagrangian filling look like?
- How many exact Lagrangian fillings does it have?

Minimum cobordisms and pinch moves

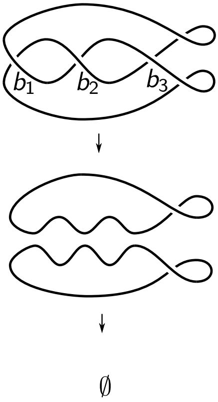
- The minimum cobordism

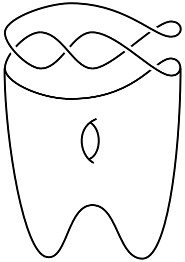
- The pinch move

Construction



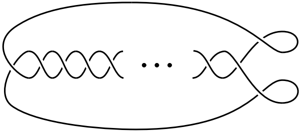
Construction





The Catalan number

The EHK construction gives Legendrian $(2, n)$ torus link



$$S_n \Big/ \left\{(\cdots,i,j,\cdots,k,\cdots)\sim(\cdots,j,i,\cdots,k,\cdots),\ \text{for any } i<k<j\right\}$$

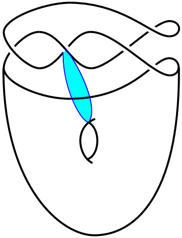
exact Lagrangian fillings.

This is called the n-th Catalan number,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Distinguish fillings

To distinguish these C_n fillings, we compute augmentations to $\mathbb{Z}_2[H_1(L)]$, which counts the homology class of holomorphic disks.



Conclusion

Theorem (P. '16)

The C_n exact Lagrangian fillings of the Legendrian $(2, n)$ torus links are of different exact Lagrangian isotopy classes.

Future directions

Augmentation Category

Constructible Sheaves Category

Future directions

Augmentation Category

Constructible Sheaves Category

Contact Topology

Algebraic Geometry

Future directions

Augmentation Category	Constructible Sheaves Category
Contact Topology	Algebraic Geometry
[EHK, '12]	[$STWZ$, '15]

Future directions

Augmentation Category	Constructible Sheaves Category
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$[EHK, '12]$	$[STWZ, '15]$
	Cluster Algebra