Polyhedra inscribed in quadrics and their geometry.

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(joint w/ J. Danciger & J.-M. Schlenker)

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A bit of history

Question (Steiner (1832))

Which graphs Γ can be obtained as 1-skeletons of a (convex) polyhedron in $\mathbb{R}^3$?

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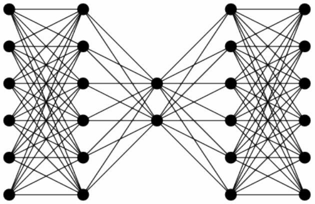
Theorem (Steinitz (1916))

Γ is the 1-skeleton of a polyhedron in $\mathbb{R}^3$ $\iff$ Γ is planar and 3–connected *(suppressing 2 vertices leaves a connected graph).*



Polyhedra inscribed in the sphere

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Which ones are inscribable in the sphere?

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Theorem (Steinitz (1927))

$\exists$ 3–*connected graphs that are not inscribable in a sphere.*

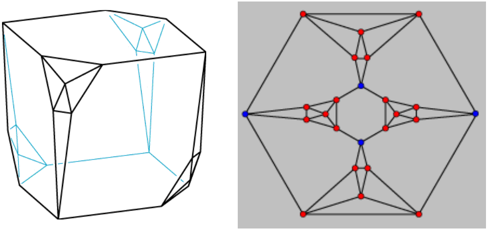

Figure: Picture by D. Eppstein and M. B. Dillencourt.

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Which 3–connected graphs are inscribable in the sphere?

Polyhedra inscribed in the sphere

Question (Steiner (1832))

Which 3–connected graphs are inscribable in the sphere?

The complete answer was given by Rivin (1992), using **hyperbolic geometry**.

Figure: Pictures by M. Grady.

Polyhedra inscribed in other quadrics

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What about other quadrics?

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Up to projective transformations, there are only 3 quadrics:

- the sphere;
- the cylinder;
- the hyperboloid.

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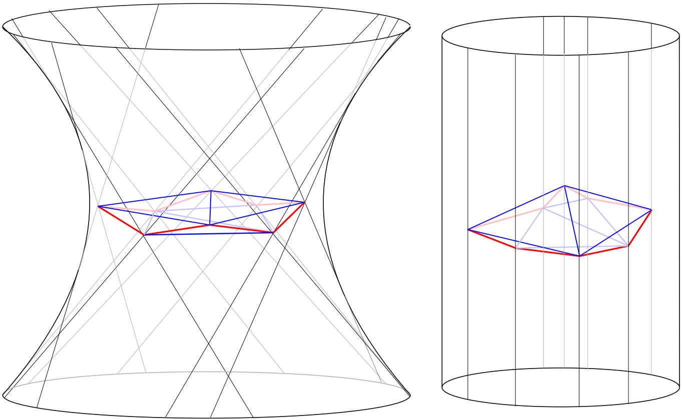
Up to projective transformations, there are only 3 quadrics:

- the sphere;
- the cylinder;
- the hyperboloid.

Jeff Danciger, Jean-Marc Schlenker and I answered, using **anti-de Sitter geometry** and **half-pipe geometry**.

Polyhedra inscribed in the cylinder and in the hyperboloid



Polyhedra inscribed in quadrics

Theorem (Danciger–M.–Schlenker (2014))

Let Γ be a planar graph. TFAE:

(C): *Γ is inscribable in the cylinder C.*

(H): *Γ is inscribable in the hyperboloid H.*

(S): *Γ is inscribable in the sphere S and Γ admits a* Hamiltonian cycle *(that is, a closed path visiting each vertex exactly once).*

Rivin (1992) characterizes when Γ is inscribable in the sphere S.

Polyhedra inscribable in the sphere, but not in the hyperboloid or cylinder

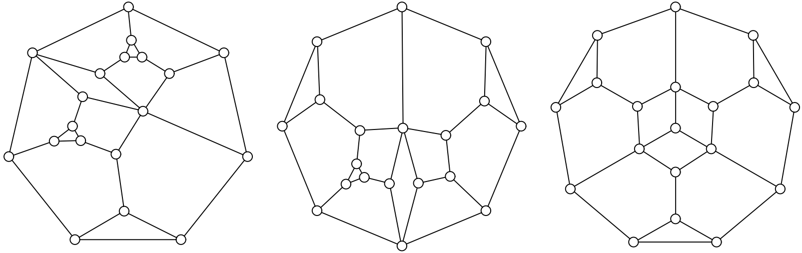


Figure: Picture by M. B. Dillencourt.

Polyhedra inscribable in the sphere, but not in the hyperboloid or cylinder

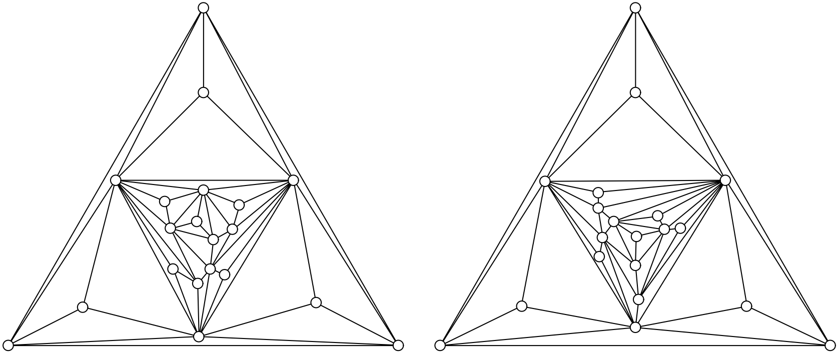


Figure: Pictures courtesy of M. B. Dillencourt.

Computational complexity

Theorem (Hodgson-Rivin-Smith (1992))

Given Γ, the problem of deciding if Γ is inscribable in a sphere is decidable in **polynomial time**.

Computational complexity

Theorem (Hodgson-Rivin-Smith (1992))

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Using Dillencourt's and our theorems, we can prove:

Corollary (Danciger-M.-Rivin-Schlenker (2014))

Given Γ, the problem of deciding if Γ is inscribable in a hyperboloid or in a cylinder is **NP-complete***.*

Hyperbolic space

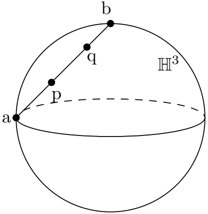
The *hyperbolic space* is the (open) unit ball

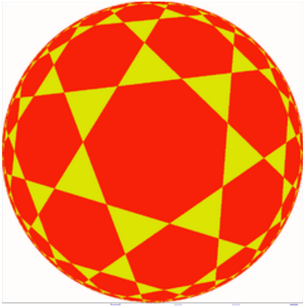
$$\mathbb{H}^3 = \{\underline{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 - x_4^2 < 0\}/\mathbb{R}^*$$

with distance

$$d(p,q) = \frac{1}{2} log \frac{|qa||bp|}{|pa||bq|}.$$

Its isometry groups is $\mathrm{PO}(3,1)$.



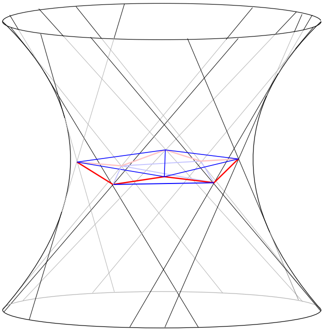


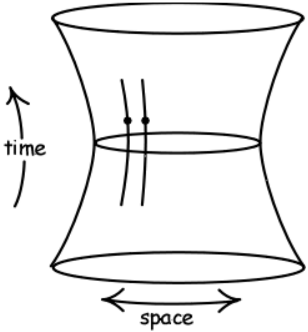
Anti-de Sitter space $\mathbb{AdS}^3$

The **anti-de Sitter space** $\mathbb{AdS}^3$ is a Lorentzian analogue of $\mathbb{H}^3$.

$$\mathbb{AdS}^3 = \{\underline{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 - x_3^2 - x_4^2 < 0\}/\mathbb{R}^*.$$

- Its isometry group is $\mathrm{PO}(2, 2)$.
- $\exists$ embeddings $\mathbb{H}^2 \hookrightarrow \mathbb{AdS}^3$.
- The faces are space-like, and the dihedral angles are in $\mathbb{R}$.

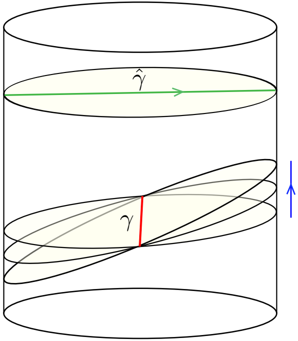


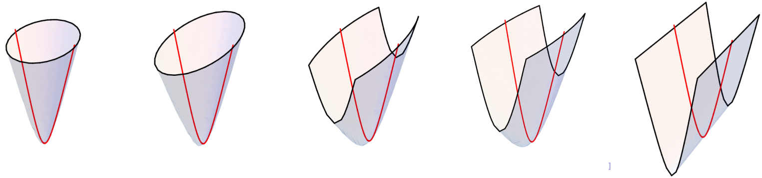


Half-pipe space $\mathbb{HP}^3$

The **half-pipe space** $\mathbb{HP}^3$:

- Limit of both $\mathbb{H}^3$ and $\mathbb{AdS}^3$.
- $\mathbb{HP}^3 = \{\underline{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 - x_4^2 < 0\}/\mathbb{R}^*$.
- $\mathbb{R}^{2,1} \rtimes \mathrm{O}(2,1)$.
- $\exists$ embeddings $\mathbb{H}^2 \hookrightarrow \mathbb{HP}^3$.
- The faces are space-like, and the dihedral angles are in $\mathbb{R}$.

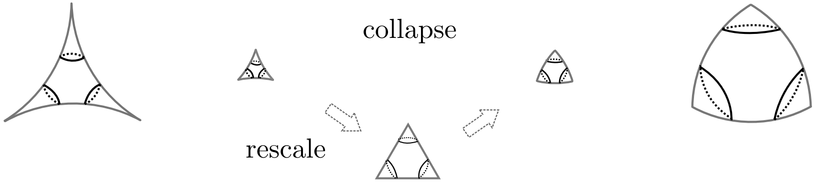




Geometric transitions

$\mathbb{H}^2$–structures collapse down to a point. After rescaling, they limit to $\mathbb{E}^2$–structures and then transition to $\mathbb{S}^2$–structures.

$(\mathbb{H}^2, \mathrm{PO}(2,1))$ $(\mathbb{R}^2, \mathbb{R}^2 \rtimes \mathrm{O}(2))$ $(\mathbb{S}^2, \mathrm{PO}(3))$



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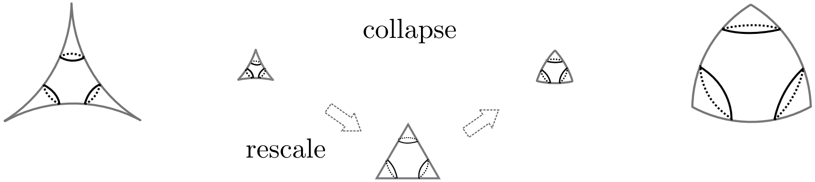
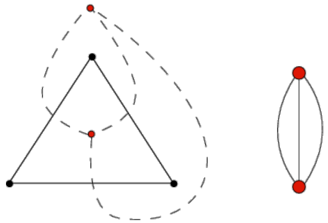


Jeff Danciger in his thesis studied a similar geometric transition from $\mathbb{H}^3$ to $\mathbb{A}\mathrm{d}\mathbb{S}^3$ structure, passing through $\mathbb{HP}^3$.

Dual of a graph

Given a planar graph $\Gamma \subset \mathbb{R}^2$, we define the *dual graph* Γ^* by:

- The vertices of Γ^* are the connected components of $\mathbb{R}^2 \setminus \Gamma$.
- The edges of Γ^* correspond to adjacent connected components.



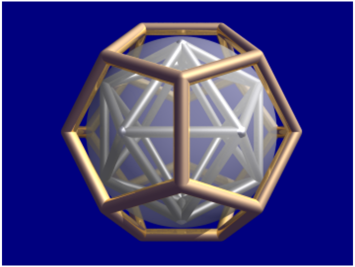


Figure: Pictures courtesy of J. Weeks (left) and M. Grady (right).

Dihedral angles in $\mathbb{H}^3$

Given a planar graph $\Gamma \subset \mathbb{R}^2$, $E(\Gamma) = \{\text{edges of } \Gamma\}$.

Let Γ^* be the graph dual to Γ. Then $E(\Gamma^*) = E(\Gamma)$

Theorem (Rivin (1992))

Let $\theta\colon\ E(\Gamma) \longrightarrow \mathbb{R}$. There is a non-planar convex ideal polyhedron in $\mathbb{H}^3$ with 1–skeleton Γ and exterior dihedral angles given by θ if and only if:

(i) $\forall e \in E(\Gamma), \theta(e) \in (0, \pi)$;

(ii) $\forall$ *cycle c in Γ^* bounding a face,* $\sum_{e\in c} \theta(e) = 2\pi$;

(iii) $\forall$ *cycle c in Γ^* not bounding a face,* $\sum_{e\in c} \theta(e) > 2\pi$.

Rivin extended a result proved by Andreev (1970) for compact and ideal polyhedra P of finite volume with dihedral angles $\leq \pi/2$.

Dihedral angles in $\mathbb{AdS}^3$ or $\mathbb{HP}^3$

Given a planar graph $\Gamma \subset \mathbb{R}^2$, $E(\Gamma) = \{\text{edges of } \Gamma\}$.

Let Γ^* be the graph dual to Γ. Then $E(\Gamma^*) = E(\Gamma)$

Theorem (Danciger-M.- Schlenker (2014))

Let $\theta\colon\ E(\Gamma) \longrightarrow \mathbb{R}$. There is a non-planar convex ideal polyhedron in $\mathbb{AdS}^3$ or $\mathbb{HP}^3$ with 1–skeleton Γ and exterior dihedral angles given by θ if and only if:

(i) *The edges on which $\theta < 0$ form a Hamiltonian cycle γ in Γ;*

(ii) *$\forall$ cycle c in Γ^* bounding a face, $\sum_{e\in c} \theta(e) = 0$;*

(iii) *$\forall$ cycle c in Γ^* not bounding a face, and containing at most two edges of γ, $\sum_{e\in c} \theta(e) > 0$.*

Induced metrics

Theorem (Rivin (1992))

Any complete hyperbolic metric of finite area on $\Sigma_{0,N}$ is induced on a unique ideal hyperbolic polyhedron (up to global isometry).

Rivin extended a result proved by Alexandrov (1944-50) for compact polyhedra.

Theorem (Danciger-M.- Schlenker)

Any complete hyperbolic metric of finite area on $\Sigma_{0,N}$ and any closed path going through each vertex exactly once are induced on a unique ideal polyhedron $P \subset \mathbb{A}\mathrm{d}\mathbb{S}^3$ (up to global isometry).

The main theorem

Theorem (Danciger–M.–Schlenker (2014))

Let Γ be a planar graph. TFAE:

(C): Γ *is inscribable in the cylinder* C.

(H): Γ *is inscribable in the hyperboloid* H.

(S): Γ *is inscribable in the sphere* S *and* Γ *admits a* Hamiltonian cycle.

Proof of (H) $\Longleftarrow$ (S)

Let P be a (convex) polyhedron inscribed in S with 1–skeleton Γ, γ be an Hamiltonian cycle, and let $\theta : E(\Gamma) \longrightarrow (0, \pi)$ be the dihedral angle map, which satisfies Rivin's conditions.

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Let P be a (convex) polyhedron inscribed in S with 1–skeleton Γ, γ be an Hamiltonian cycle, and let $\theta : E(\Gamma) \longrightarrow (0, \pi)$ be the dihedral angle map, which satisfies Rivin's conditions.
We define $\theta' : E(\Gamma) \longrightarrow \mathbb{R}_{\neq 0}$ by

$$\theta'(e) = \begin{cases} \theta(e) & \text{if } e \nsubseteq \gamma \\ \theta(e) - \pi & \text{if } e \subseteq \gamma \end{cases}$$

Then θ' satisfies our conditions, so P can be inscribed in H.

Statement of the theorem

Theorem (Danciger-M.- Schlenker (2014))

Given $\theta\colon\ E(\Gamma) \longrightarrow \mathbb{R}$, $\exists$ *an ideal polyhedron in* $\mathbb{AdS}^3$ *or* $\mathbb{HP}^3$ *with* 1–*skeleton* Γ *and exterior dihedral angles given by* θ *if and only if:*

(i) *The edges on which* $\theta < 0$ *form a Hamiltonian cycle* γ *in* Γ*;*

(ii) $\forall$ *cycle* c *in* Γ^* *bounding a face,* $\sum_{e\in c}\theta(e) = 0$*;*

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Theorem (Danciger-M.- Schlenker (2014))

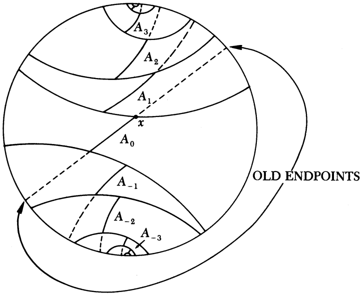
The following maps are homeo:

- $\Psi_{HP} : \mathrm{HPPoly}_N \longrightarrow \mathcal{A}$
- $\Phi : \overline{\mathrm{AdSPoly}}_N = \mathrm{AdSPoly}_N \cup \mathrm{polyg}_N \longrightarrow \mathcal{T}(\Sigma_{0,N})$
- $\Psi_{AdS} : \mathrm{AdSPoly}_N \longrightarrow \mathcal{A}$

Tools the proof

Earthquakes and bending:

- $P \in \mathrm{AdSPoly}_N \rightsquigarrow p_L, p_R \in \mathrm{polyg}_N \rightsquigarrow m_L, m_R \in \mathcal{T}(\Sigma_{0,N})$;
- m_L, m_R determines P w/ bending $\theta \in \mathbb{R}^E \iff m_L = E_{2\theta} m_R$.



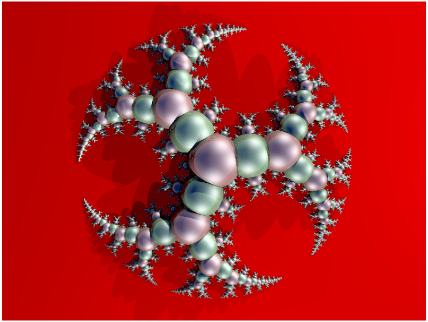


Figure: Pictures courtesy of S. Kerckhoff and Y. Kabaya.

Sketch of the proof (continuation...)

1. Ψ_{HP} is a homeo:
   - $P \in \mathrm{HPPoly}_N \rightsquigarrow (p, V) \rightsquigarrow (m, W)$;
   - Given θ, solve for p by minimizing a length function.

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Sketch of the proof (continuation...)

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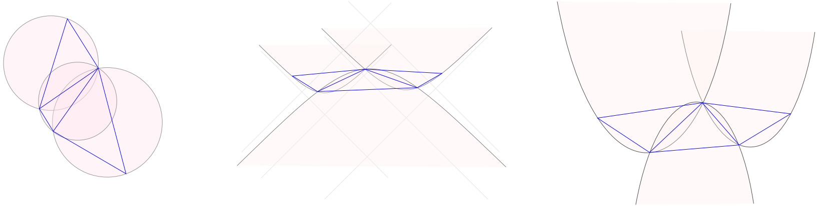
2. Φ is a homeo:
   - Φ proper (direct proof);
   - Φ is a local homeo (use Pogolorov map);
3. Ψ_{AdS} is a homeo:
   - Ψ_{AdS} proper (direct proof);
   - Ψ_{AdS} is a local homeo (duality b/ metric and angle data);

Exotic Delaunay traingulations

Exotic Delaunay traingulations (w/ J. Danciger & J.-M. Schlenker)



Euclidean space $\mathbb{E}^2$: Circles

Minkowski space $\mathbb{R}^{1,1}$: Hyperbolas

'Limit space' $\mathbb{R}^{1,0,1}$: Parabolas

Theorem (Danciger-M.-Schlenker)

For any quadratic form Q on $\mathbb{R}^d$ and for any finite set $X \subset \mathbb{R}^d$, $\exists$ a unique Q-Delaunay triangulation of $\mathrm{CH}(X)$.

Bending conjecture

$\mathrm{QF}(\Sigma) \subset \{\text{hyp str on } \Sigma \times \mathbb{R}\}$.

Theorem (Bers)

$\mathrm{QF}(\Sigma) \cong \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$.

Conjecture (Bending in $\mathbb{H}^3$)

$\mathrm{QF}(\Sigma) \cong \mathcal{ML}(\Sigma) \times \mathcal{ML}(\Sigma)$.

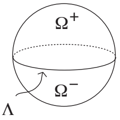
$\mathrm{GH}(\Sigma) \subset \{\text{AdS str on } \Sigma \times \mathbb{R}\}$.

Theorem (Mess)

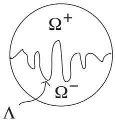
$\mathrm{GH}(\Sigma) \cong \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$.

Conjecture (Bending in $\mathbb{A}\mathrm{d}\mathbb{S}^3$)

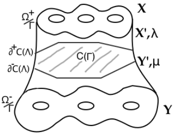
$\mathrm{GH}(\Sigma) \cong \mathcal{ML}(\Sigma) \times \mathcal{ML}(\Sigma)$.

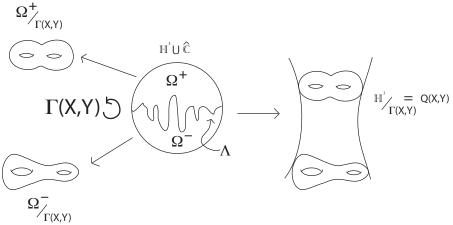


Fuchsian Case



quasi-Fuchsian Case





End

Proof of (H) $\Longrightarrow$ (S)

Let P be a (convex) polyhedron inscribed in H. Let $\theta : E(\Gamma) \longrightarrow \mathbb{R}_{\neq 0}$ be the dihedral angle map which satisfies our conditions, and let γ be the cycle of its 'negative' edges..

Proof of (H) $\Longrightarrow$ (S)

Let P be a (convex) polyhedron inscribed in H. Let $\theta : E(\Gamma) \longrightarrow \mathbb{R}_{\neq 0}$ be the dihedral angle map which satisfies our conditions, and let γ be the cycle of its 'negative' edges..
We can choose $t > 0$ s.t.

- $\forall e \in E(\Gamma)$, $t\theta(e) \in (-\pi, \pi)$;
- $\forall$ cycle c in Γ^* not bounding a face, then the sum of the values of $t\theta$ on the edges of c is $> -\pi$.

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Let $\theta' : E(\Gamma) \longrightarrow (0, \pi)$ be defined by

$$\theta'(e) = \begin{cases} t\theta(e) & \text{if } e \not\subseteq \gamma \\ \pi + t\theta(e) & \text{if } e \subseteq \gamma \end{cases}$$

Then θ' satisfies Rivin's conditions. Therefore P be a (convex) polyhedron inscribed in S.