A Quantitative Look at Lagrangian Cobordisms

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Lagrangians and Legendrians

Symplectic Manifold \((X^{2n}, \omega)\)

Lagrangian Submanifold
\(L^n : \omega|_T L \equiv 0\)

Contact Manifold \((Y^{2n+1}, \xi)\)

Legendrian Submanifold
\(\Lambda^n : T\Lambda \subset \xi\)
Lagrangians and Legendrians

Symplectic Manifold \((X^{2n}, \omega)\)

- Exact Symplectic: \(\omega = d\lambda\)

Lagrangian Submanifold
\(L^n : \omega\mid_{TL} \equiv 0\)

- Exact Lagrangian: \(\lambda = df\)

Contact Manifold \((Y^{2n+1}, \xi)\)

Legendrian Submanifold
\(\Lambda^n : T\Lambda \subset \xi\)
The Symplectization of a Contact Manifold

**Standard Contact Manifold:** \(( \mathbb{R}^{2n+1}, \ker \alpha )\)

\[ J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}, \quad \alpha = dz - \sum_i y_i dx_i \]
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**Symplectization:** \((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^s \alpha))\)

There are no closed, exact Lagrangians (Gromov); For a Legendrian \(\Lambda\), the cylinder \(\mathbb{R} \times \Lambda\) is an exact Lagrangian.
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- There are no closed, exact Lagrangians (Gromov);
- For a Legendrian \(\Lambda\), the cylinder \(\mathbb{R} \times \Lambda\) is an exact Lagrangian.
A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$ means:

$$\mathbb{R}^{2n+2}$$

$\Lambda_-$ and $\Lambda_+$ are Legendrian submanifolds in $\{s = s_\pm\}$; $L$ is Lagrangian and cylindrical over $\Lambda_\pm$ at $\pm \infty$:

$L = \mathbb{R} \times \Lambda_\pm$ outside $[s_- , s_+]$;

$L$ is embedded and exact:

$$e_s \alpha \big|_L = df, f = \text{constant} \pm \text{outside} [s_-, s_+]$.

Arise in relative SFT (Eliashberg-Givental-Hofer)

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A Lagrangian cobordism from \( \Lambda_- \) to \( \Lambda_+ \) means:

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  \[
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Arise in relative SFT (Eliashberg-Givental-Hofer)
Qualitative Questions

- Given $\Lambda_-, \Lambda_+ \subset \mathbb{R}^{2n+1}$, does there exist a Lagrangian cobordism between them?
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\[ \exists \quad \nexists \]

Non-symmetric relation!
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- How topologically rigid are Lagrangian cobordisms?

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Fillings realize 4-ball genus!
Qualitative Questions

- Given $\Lambda_- , \Lambda_+ \subset \mathbb{R}^{2n+1}$, does there exist a Lagrangian cobordism between them?

- How topologically rigid are Lagrangian cobordisms?

A variety of qualitative questions have been studied by: Chantraine, Ekholm, Honda, Kálmán, Dimitroglou Rizell, Ghiggini, Golovko, Cornwell, Ng, Sivek, Bourgeois, Sabloff, Traynor, Capovilla-Searle, Hayden, Pan, ...
(Length) Given $\Lambda_-, \Lambda_+ \subset \mathbb{R}^{2n+1}$, what is the minimal "length" of any cobordism between them?

Given a Lagrangian cobordism, what is its "width"?

\[ h = s_+ \quad \quad 0 = s_- \]
Quantitative Questions

- **(Length)** Given $\Lambda_-, \Lambda_+ \subset \mathbb{R}^{2n+1}$, what is the minimal "length" of any cobordism between them?

  \[ h = s_+ - s_- \]

- **(Width)** Given a Lagrangian cobordism, what is its "width"?

  ![Diagram of a Lagrangian cobordism](image-url)
Constructions of Lagrangian Cobordisms
Outline

1. Constructions of Lagrangian Cobordisms
2. Length of a Lagrangian cobordism
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1. Constructions of Lagrangian Cobordisms

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Isotopy Lemma (Eliashberg, Chantraine, Golovko, Ekholm-Honda-Kálmán, ...) 

Suppose $\Lambda_-$ and $\Lambda_+$ are Legendrian isotopic. Then there exists a Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$. 

Remark: The Lagrangian is not the trace of the isotopy. Most slices of the Lagrangian will not be Legendrian.
Isotopy Lemma (Eliashberg, Chantraine, Golovko, Ekholm-Honda-Kálmán, ...)

Suppose $\Lambda_-$ and $\Lambda_+$ are Legendrian isotopic. Then there exists a Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$.

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Qualitatively Symmetric Concordances:

\[ \Lambda_+ \quad \Lambda_- \]

\[ \exists \quad \exists \]

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Theorem (Dimitroglou Rizell, Ekholm-Honda-Kálmán, Bourgeois-Sabloff-T)

If \( \Lambda^+ \) is obtained from \( \Lambda^- \) by a “cusp-surgery”, then there exists a Lagrangian cobordism from \( \Lambda^- \) to \( \Lambda^+ \).
Theorem (Dimitroglou Rizell, Ekholm-Honda-Kálmán, Bourgeois-Sabloff-T)

If $\Lambda_+$ is obtained from $\Lambda_-$ by a "cusp-surgery", 

\begin{center}
\includegraphics[width=0.5\textwidth]{cusp-surgery-diagram.png}
\end{center}
Theorem (Dimitroglou Rizell, Ekholm-Honda-Kálmán, Bourgeois-Sabloff-T)

If $\Lambda_+$ is obtained from $\Lambda_-$ by a "cusp-surgery",

then there exists a Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$. 

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Construction Example

Lagrangian genus 1 filling of a Legendrian $m(5_2)$:

Legendrian isotopy and cusp pinches as you move up!
1. Constructions of Lagrangian Cobordisms

2. Length of a Lagrangian cobordism

3. Width of a Lagrangian Cobordism
**Question:** Given $\Lambda_-, \Lambda_+ \subset \mathbb{R}^{2n+1}$, what is the “minimal length" of any cobordism between them?

$$h = s_+ \quad \rightarrow \quad 0 = s_- \quad \rightarrow$$

minimal length $= \inf \{ h : \exists \text{ Lagrangian cobordism from } \Lambda_- \text{ to } \Lambda_+ \text{ that is cylindrical outside } [0, h] \}$. 
Theorem (Sabloff-T, ’16: Selecta Mathematica)

There exists an arbitrarily short Lagrangian cobordism between

- a Legendrian and its vertical translate,
- a Legendrian and its horizontal translate,
- a Legendrian and its vertical expansion.
Flexibility

Theorem (Sabloff-T, '16: *Selecta Mathematica*)

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Theorem (Sabloff-T, ’16)

There exist obstructions to arbitrarily short Lagrangian cobordisms between

1. a Legendrian and its vertical contraction;

\[ h \sim \ln 2 \]

\[ s \]

\[ 0 \]

\[ \Lambda_+ \]

\[ \Lambda_- \]
Theorem (Sabloff-T, ’16)

There exist obstructions to arbitrarily short Lagrangian cobordisms between

1. a Legendrian and its vertical contraction;

\[ h \sim \ln 2 \]

2. vertically shifted Hopf links:

\[ h \sim \begin{cases} 
\ln \left( \frac{1 - u}{1 - v} \right), & \text{if } u \leq v, \\
\ln \left( \frac{u}{v} \right), & \text{if } u \geq v.
\end{cases} \]
(Step 1) Assign "capacities" to a Legendrian

\[ \mathbf{c}(\Lambda, \varepsilon, \theta) \in \mathbb{R}_{>0} \cup \{\infty\}, \]

\(\varepsilon\) is an augmentation of the DGA \(\mathcal{A}(\Lambda), \varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}_2, 0)\),
\(\theta \in LCH^*(\Lambda, \varepsilon)\).
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Example:

\[ \exists 0 \neq \lambda \in LCH^1(U(r), \varepsilon); \quad c(U(r), \varepsilon, \lambda) = r. \]

Fundamental Class         Fundamental Capacity
(Step 1) Assign “capacities” to a Legendrian

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Example:

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Fundamental Class \quad Fundamental Capacity

For \( \theta \neq 0 \), \( c(\Lambda, \varepsilon, \theta) \) is always the height of a Reeb chord!
(Step 2) From $\varepsilon_-, \theta_-$ for $\Lambda_-$ and Lagrangian cobordism $L$ from $\Lambda_-$ to $\Lambda_+$, get induced $\varepsilon_+, \theta_+$ for $\Lambda_+$. 
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[Ekholm-Honda-Kálmán]

\[
\begin{array}{ccc}
\Lambda_+ & \xrightarrow{\sim} & \mathcal{A}(\Lambda_+) \\
\downarrow^{\Phi(L)} & & \downarrow^{\varepsilon_+} \\
\Lambda_- & \xrightarrow{\sim} & \mathcal{A}(\Lambda_-) \xrightarrow{\varepsilon_-} \mathbb{F}_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
LCH^*(\Lambda_+, \varepsilon_+) & \xrightarrow{\psi_{L,\varepsilon_-}} & \mathbb{F}_2 \\
\downarrow^{\theta_+} & & \downarrow^{\theta_-} \\
LCH^*(\Lambda_-, \varepsilon_-) & & \\
\end{array}
\]

Question: How do capacities $c(\Lambda_+, \varepsilon_+, \theta_+)$ and $c(\Lambda_-, \varepsilon_-, \theta_-)$ compare?
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[Ekholm-Honda-Kálmán]

\[
\begin{array}{ccc}
\Lambda_+ & \sim & A(\Lambda_+) \\
\downarrow \phi(L) & \searrow \varepsilon_+ & \\
\Lambda_- & \sim & A(\Lambda_-) \rightarrow \mathbb{F}_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
LCH^*(\Lambda_+, \varepsilon_+) & \rightarrow & \theta_+ \\
\downarrow \psi_{L,\varepsilon_-} & \nearrow & \\
LCH^*(\Lambda_-, \varepsilon_-) & \rightarrow & \theta_- \\
\end{array}
\]

**Question:** How do capacities $c(\Lambda_+, \varepsilon_+, \theta_+)$ and $c(\Lambda_-, \varepsilon_-, \theta_-)$ compare?
Lower Bound to Length

(Step 3) Relate capacities for ends of a Lagrangian cobordism.

Length-Capacity Inequality (Sabloff-T)

If $L$ is a Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$ that is cylindrical outside $[0, h]$, then

$$e^0 c(\Lambda_-, \varepsilon_-, \theta_-) \leq e^h c(\Lambda_+, \varepsilon_+, \theta_+)$$
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Remember $\varepsilon_+, \theta_+$ are induced by $L$. 
Lower Bound to Length

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Remember $\varepsilon_+, \theta_+$ are induced by $L$.

Get lower bounds to length of a cobordism!

$$\ln \left( \frac{c(\Lambda_-, \varepsilon_-, \theta_-)}{c(\Lambda_+, \varepsilon_+, \theta_+)} \right) \leq h.$$
By Length-Capacity Inequality:

\[
\ln \left( \frac{2}{1} \right) = \ln \left( \frac{c(U(2)), \varepsilon_-, \lambda_-}{c(U(1)), \varepsilon_+, \lambda_+} \right) = \ln \left( \frac{c(\Lambda_-, \varepsilon_-, \lambda_-)}{c(\Lambda_+, \varepsilon_+, \lambda_+)} \right) \leq h.
\]
By Length-Capacity Inequality:

$$\ln \left( \frac{2}{1} \right) = \ln \left( \frac{c(U(2))}{c(U(1))} , \varepsilon_- , \lambda_- \right) = \ln \left( \frac{c(\Lambda_- , \varepsilon_- , \lambda_- )}{c(\Lambda_+ , \varepsilon_+ , \lambda_+ )} \right) \leq h. $$

**Question:** Can we get arbitrarily close to $h = \ln 2$?
By Length-Capacity Inequality:

\[
\ln \left( \frac{2}{1} \right) = \ln \left( \frac{c(U(2)), \varepsilon_-, \lambda_-}{c(U(1)), \varepsilon_+, \lambda_+} \right) = \ln \left( \frac{c(\Lambda_-, \varepsilon_-, \lambda_-)}{c(\Lambda_+, \varepsilon_+, \lambda_+)} \right) \leq h. 
\]

**Question:** Can we get arbitrarily close to \( h = \ln 2 \)?

**Answer:** Yes!
exists Lagrangian cobordism from $\Lambda_-=U(2)$ to $\Lambda_+=U(1)$ of length $A$:
∃ Lagrangian cobordism from $\Lambda_- = U(2)$ to $\Lambda_+ = U(1)$ of length $A$:

Legendrian isotopy: $\lambda_s(t) = (x(t), \rho(s)y(t), \rho(s)z(t))$

So, $\exists$ embedded Lagrangian cobordism when $1 < \frac{e^A}{2} \iff \ln 2 < A$. 
∃ Lagrangian cobordism from \( \Lambda_- = U(2) \) to \( \Lambda_+ = U(1) \) of length \( A \):

Legendrian isotopy:
\[
\lambda_s(t) = (x(t), \rho(s)y(t), \rho(s)z(t))
\]

Lagrangian immersion:
\[
\Gamma(s, t) = (s, x(t), \rho(s)y(t), \rho(s)z(t) + \rho'(s)z(t))
\]
∃ Lagrangian cobordism from $\Lambda_- = U(2)$ to $\Lambda_+ = U(1)$ of length $A$:

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Embedding condition: $\frac{d}{ds}(e^{s}\rho(s)) \neq 0$
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So,

∃ embedded Lagrangian cobordism when $1 < e^{A/2} \iff \ln 2 < A$. 
1. Constructions of Lagrangian Cobordisms
2. Length of a Lagrangian cobordism
3. Width of a Lagrangian Cobordism
Width of a Symplectic Manifold

\[ B^{2n}(c) := \left\{ (x_1, y_1, \ldots, x_n, y_n) : \pi \sum_i (x_i^2 + y_i^2) \leq c \right\} \subset (\mathbb{R}^{2n}, \omega_0). \]
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Width of a symplectic manifold \((X, \omega)\):

$$w(X) := \sup \{ c : \exists \psi : B^{2n}(c) \to X, \psi^* \omega = \omega_0 \}.$$
Width of a Symplectic Manifold

\[ B^{2n}(c) := \left\{ (x_1, y_1, \ldots, x_n, y_n) : \pi \sum_i (x_i^2 + y_i^2) \leq c \right\} \subset (\mathbb{R}^{2n}, \omega_0). \]

Width of a symplectic manifold \((X, \omega)\):

\[ w(X) := \sup\{ c : \exists \psi : B^{2n}(c) \to X, \psi^*\omega = \omega_0 \}. \]

We are working in \( X = \mathbb{R} \times J^1 M \): \( w(\mathbb{R} \times J^1 M) = \infty \).
Given a Lagrangian submanifold $L \subset (X, \omega)$, the relative width is:

$$w(X, L) = \sup \left\{ c \mid \exists \psi : B^{2n}(c) \to X, \psi^*\omega = \omega_0, \psi^{-1}(L) = B^{2n}(c) \cap \mathbb{R}^n \right\}.$$
Given a Lagrangian submanifold $L \subset (X, \omega)$, relative width is:

$$w(X, L) = \sup \left\{ c \mid \exists \psi : B^{2n}(c) \to X, \psi^* \omega = \omega_0, \psi^{-1}(L) = B^{2n}(c) \cap \mathbb{R}^n \right\}.$$
Given a Lagrangian cobordism $L$, for $-\infty \leq a < b \leq \infty$, 

$$L^b_a := \{(s, x, y, z) \in L : a < s < b\} \subset (a, b) \times J^1M.$$
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**Question:** Can we calculate $w(L^b_a)$, for some $L, a, b$?
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**Question:** Can we calculate $w(L^b_a)$, for some $L, a, b$?

**Answer:** Yes!
Lemma

For any Lagrangian cobordism $L$, for any $a$,

$$w \left( L_a^{\infty} \right) = \infty.$$
Lemma

For any Lagrangian cobordism $L$, for any $a$,

$$w(L^\infty_a) = \infty.$$  

Proof Sketch:
Lemma

For any Lagrangian cobordism $L$, for any $a$,

$$w(L^\infty_a) = \infty.$$ 

Proof Sketch:

[Diagram showing the concept of infinite width with labeled regions B(R) and B(r).]
Lemma

For any Lagrangian cobordism \( L \), for any \( a \),

\[
    w \left( L^\infty_a \right) = \infty.
\]

Proof Sketch:

Chop off top! We will consider:

\[
    a = -\infty, \quad s_+ \leq b < +\infty.
\]
Cylindrical Lagrangian Cobordisms: $L = \mathbb{R} \times \Lambda$, for a Legendrian $\Lambda$. 

Question: Can we calculate $w((\mathbb{R} \times \Lambda)_{b-\infty})$ for some $\Lambda$ and for some $b$?

Suffices to understand $b = 0$:

**Lemma**

For any Legendrian $\Lambda$, $w((\mathbb{R} \times \Lambda)_{b-\infty}) = e^{bw((\mathbb{R} \times \Lambda)_0-\infty)}$.

Question: Can we calculate $w((\mathbb{R} \times \Lambda)_0-\infty)$ for some $\Lambda$?

Answer: Yes!
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**Lemma**

*For any Legendrian $\Lambda$,*

$$w \left( (\mathbb{R} \times \Lambda)^b_{-\infty} \right) = e^b w \left( (\mathbb{R} \times \Lambda)^0_{-\infty} \right).$$
Cylindrical Lagrangian Cobordisms: $L = \mathbb{R} \times \Lambda$, for a Legendrian $\Lambda$.

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Cylindrical Lagrangian Cobordisms: \( L = \mathbb{R} \times \Lambda \), for a Legendrian \( \Lambda \).

**Question:** Can we calculate \( w((\mathbb{R} \times \Lambda)_b)_{-\infty} \) for some \( \Lambda \) and for some \( b \)?

Suffices to understand \( b = 0 \):

**Lemma**

*For any Legendrian \( \Lambda \),

\[
w((\mathbb{R} \times \Lambda)^b_{-\infty}) = e^b w((\mathbb{R} \times \Lambda)^0_{-\infty}).
\]

**Question:** Can we calculate \( w((\mathbb{R} \times \Lambda)^0_{-\infty}) \) for some \( \Lambda \)?

**Answer:** Yes!
Theorem (Sabloff-T): 
\[ (R \times U(r))_{0-\infty} = 2r. \]
Width of Cylinder over Legendrian Unknot

Theorem (Sabloff-T)

\[ w\left((\mathbb{R} \times U(r))_{-\infty}^0\right) = 2r. \]
$w \left( (\mathbb{R} \times U(r))^0_{-\infty} \right) \leq 2r$ follows from:

**Theorem (Sabloff-T)**

Suppose $\Lambda$ is a Legendrian that admits an augmentation. Then

$$w \left( (\mathbb{R} \times \Lambda)^0_{-\infty} \right) \leq 2c(\Lambda),$$

where $c(\Lambda)$ is the minimum fundamental capacity (for any augmentation).
Upperbound to Width of a Legendrian

Theorem (Sabloff-T)

Suppose \( \Lambda \) is a Legendrian that admits an augmentation. Then

\[
\text{size of ball } \leq 2 \text{ “fundamental Reeb chord height” in } \partial = \Lambda
\]

where \( c(\Lambda) \) is the minimum fundamental capacity (for any augmentation).
Obstructions to Embeddings:

Proof Sketch:

- Suppose there is an embedding $\psi$ of $B(\alpha)$. 

Classical Isoperimetric Inequality shows $\alpha \leq 2B \leq 2c(\Lambda, \varepsilon, \lambda)$. 

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Lagrangian Cobordisms
Tech Topology
Proof Sketch:

- Suppose there is an embedding $\psi$ of $B(\alpha)$.
- By property of the fundamental class $\lambda \in LCH^*(\Lambda, \varepsilon)$, through $\psi(0) \in L$ there is a $J$-holomorphic "disk" of area $A \leq c(\Lambda, \varepsilon, \lambda)$. 

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Lisa Traynor (Bryn Mawr)
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2r \leq w \left( (\mathbb{R} \times U(r))^0_{-\infty} \right) \text{ follows from:}

**Theorem (Sabloff-T)**

Suppose $\Lambda$ has a “vertically extendable” Reeb chord of height $r$. Then

$$2r \leq w \left( (\mathbb{R} \times \Lambda)^0_{-\infty} \right).$$
Lowerbound to width of a Legendrian

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Vertically Extendable

No \quad Yes
Constructing Embeddings

Proof Sketch:

$$\psi : \mathbb{R} \times J^1 M \rightarrow T^*\mathbb{R}_+ \times T^* M$$

$$(s, x, y, z) \mapsto ((e^s, z), (x, e^s y))$$
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What if $L$ is a non-cylindrical cobordism?

Example:

Question: Can we still find a symplectic embedding of $B(2)$?

Answer: Yes!

Question: Can we embed a bigger ball?

Answer: No!

Width does not see the negative end!

Lisa Traynor (Bryn Mawr)
What if $L$ is a **non-cylindrical cobordism**?

**Example:**

- If $L$ is a non-cylindrical cobordism, the width might not be as simple as in the cylindrical case. For example, consider a cobordism with varying widths over time, such as $1 \rightarrow 2 \rightarrow 1$.

**Question:** Can we still find a symplectic embedding of $B(2)$?

**Answer:** Yes! However, if we try to embed a ball $B(3)$, it may not be possible due to the varying widths of the cobordism.

**Width does not see the negative end!**
What if $L$ is a non-cylindrical cobordism?

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**Example:**

```
1 0
.5
```

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Width of Non-Cylindrical Lagrangian Cobordisms

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**Example:**

![Diagram of non-cylindrical cobordism](image)

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Theorem (Sabloff-T)

If $L$ is a Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$ and $\Lambda_-$ is fillable, then

$$w\left(L^0_{-\infty}\right) \leq 2c(\Lambda_+),$$

where $c(\Lambda_+)$ is the minimum fundamental capacity (for any augmentation).
Theorem (Sabloff-T)

If $L$ is a Lagrangian cobordism from $\Lambda^-$ to $\Lambda^+$ and $\Lambda^-$ is fillable, then

$$w \left( L^0_{-\infty} \right) \leq 2c(\Lambda^+),$$

where $c(\Lambda^+)$ is the minimum fundamental capacity (for any augmentation).

Proof is similar in spirit to the proof when $L = \mathbb{R} \times \Lambda$:

Use Seidel Isomorphism to get the existence of a $J$-holomorphic disk through $\psi(0) \in \psi(B(\alpha))$. 
Can reprove our earlier length result between $\Lambda_- = U(2)$ and $\Lambda_+ = U(1)$: $h \geq \ln 2$
Length-Width Connection

Can reprove our earlier length result between $\Lambda_- = U(2)$ and $\Lambda_+ = U(1)$: $h \geq \ln 2$

Corollary

Suppose $L$ is a Lagrangian cobordism from $\Lambda_- = U(2)$ to $\Lambda_+ = U(1)$ that is cylindrical outside $[-h, 0]$. Then

$$\ln 2 = \ln \left( \frac{2}{1} \right) = \ln \left( \frac{c(U(2))}{c(U(1))} \right) \leq h.$$
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Proof:

$$2e^{-h}c(U(2)) = e^{-h}w((\mathbb{R} \times \Lambda_-)^0_{-\infty}) = w((\mathbb{R} \times \Lambda_-)^{-h}_{-\infty}) \leq w(L^0_{-\infty}) \leq 2c(U(1))$$
Much to be understood about length and width!
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- Calculate widths of other Lagrangian cobordisms when $\Lambda_-$ admits an augmentation/filling!
Questions

Much to be understood about length and width!

- Calculate widths of other Lagrangian cobordisms when $\Lambda -$ admits an augmentation/filling!

- Can we calculate the width or length of a Lagrangian cobordism when $\Lambda -$ does not admit an augmentation/filling?
  
  For example, when $\Lambda -$ is stabilized or loose?
Much to be understood about length and width!

- Calculate widths of other Lagrangian cobordisms when $\Lambda_-$ admits an augmentation/filling!
- Can we calculate the width or length of a Lagrangian cobordism when $\Lambda_-$ does *not* admit an augmentation/filling?
  
  For example, when $\Lambda_-$ is *stabilized* or *loose*?

Thank you!