Braid index, the fractional Dehn twist coefficients and Upsilon

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A knot is a smooth embedding $S^1 \to S^3$
A link is a smooth embedding $S^1 \times \mathbb{R} \to S^3$

Braids are representatives of knots/links

Braid group on $n$ strands

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2 \rangle$$

$$\sigma_1 \sigma_2^{-1} \in B_3$$

closure

The closure of any braid is a knot/link

Alexander (1923): Every knot or link can be realized as the closure of some braid (in fact infinitely many)

ex

Defn: the braid index of a knot or link $K$ is the minimal $n$ so that $K$ is the closure of an $n$-braid

A way to measure the "complexity" of a knot

Question: What is known about the braid index?

Answer: Not a lot

A famous result: Morton-Franks-Williams inequality '87

let $d_+, d_-$ the max/min degree in $v$ in HOMFLY poly of $K$

the braid index $b_K$ of $K$ satisfies
\( \frac{1}{2} (d_+ - d_-) + 1 \leq b_k \)

Jones '87: On all but 5 of the knots up to 10 crossings this is sharp

Kawamura '06: Fails to be sharp for so many & the defect can be arbitrarily large

Corollary of MFJ:

if a knot \( K \) has an \( n \)-braid rep \( \beta \) where \( \beta = \Delta^2 \alpha \) then \( K \) has braid index \( n \)

1. special braid

"full twist" \( (\sigma_1 \cdots \sigma_{n-1})^n \)

Braids as mapping classes

\( B_n \) is the mapping class group of \( n \)-punctured disk \( D_n \)

\( \rightarrow \) group of orientation preserving homeos of \( D_n \) that fix \( \partial D_n \) isotopically

that fix \( \partial D_n \)

every element \( \phi \) in \( \text{MC}_G \) of a surface with one boundary component can be assigned its \text{fractional Dehn twist coeff} (FDTC)

that measures the amount of twisting \( \phi \) effects about the boundary

\( \begin{align*}
\text{ex: in } B_3 & \quad \tau(\Delta^2) = 1 \\
& \quad \tau(\sigma_1 \sigma_2) = \frac{1}{3} \\
& \quad \tau(\sigma_1) = 0 \\
& \quad \tau(\sigma_1 \sigma_2^{-1}) = 0
\end{align*} \)

Idea of def^2: compactify the universal cover \( \bar{D}_n \) of \( D_n \) to get \( \bar{D}_n \) use the action of the lift of \( \beta \) on \( \partial \bar{D}_n \) to construct a map \( \Theta : B_n \to \text{Homeo}^+(\mathbb{R}) \)

define \( \tau(\beta) = \text{translation number of } \Theta(\beta) \)
Theorem (Feller-H.)

Fix \( n \geq 2 \), any \( n \)-braid \( \beta \) with \( |T(\beta)| > \alpha - 1 \) realizes the braid index of its closure.

This is close to optimal: the bound could at best be improved to \( n - 2 \), there exists an \( n \)-braid \( \beta \) with \( \text{FDTC} \) \( n - 2 \) that do not realize the braid index of their closures (Malyutin-Netsvetaev).

Recall for MFV: \( \beta = \Delta^2 \alpha \)

\( \alpha \) positive

Example: \( \beta = (\Delta^2)^{h}(\sigma_1 \sigma_2)^{-k} \in \mathcal{B}_4 \)

\( h \) arbitrarily large \( \tau (\beta) = 4 \)

Tools that go into the proof:

- A left-order on \( \mathcal{B}_n \) (due to Dehornoy)
- A characterization of FDTC in terms of order (Malyutin)
- A calculation of \( \Upsilon \), a knot concordance invariant due to Ozsváth-Stipsicz-Szabó for torus knots
- A characterization of FDTC in terms of \( \Upsilon \), a braid quasimorphism arising from \( \Upsilon \)
- Generalized Jones conjecture (Brandenburgy)

Due to Dynnikov-Prasolov, Lafontaine-Menâšco...

(2013) Consider \( n \)-braid \( \beta \), \( m \)-braid \( \alpha \)

\( \beta = 2 \) and \( m \) is the braid index of \( \beta \)

Then \( |\text{wr}(\beta) - \text{wr}(\alpha)| \leq n - m \)

Exponent sum.
A left order on \(B_n\) (Dehornoy)

\[ \beta > 1 \text{ if it can be written as a } \sigma_i\text{-positive word for some } i \]

\(\Rightarrow\) no \(\sigma_i\)'s up to \(i\)

\(\sigma_1\) only appears positively

\[0_2 > 1, 0_3^{-1000} > 1, 0_3^{-1000} 0_2 > 1\]

\(\alpha < \beta \text{ if } \alpha - \beta > 1\)

for any \(\beta, \exists! m \text{ s.t. } (\Delta^2)^{m+1} \beta \geq (\Delta^2)^m \]

\[m = \left\lfloor \frac{\beta}{\alpha} \right\rfloor\]

Malyutin '04

\[\tau(\beta) = \lim_{k \to \infty} \frac{4\beta k^3}{k}\]

**Thm (Feller - H):**

Fix \(n \geq 2\). If an \(n\)-braid \(\beta\) satisfies \(\beta \geq \Delta^{2n}\) or \(\beta \leq \Delta^{-2n}\) then \(\beta\) realizes braid index of its closure

**Concordance:** Two links \(K, L\) are concordant if \(\exists\) ori. smooth embedding of disjoint annuli in \(S^3 \times [0, 1]\) s.t. ori. boundary is \(K \times \{0\} \cup L \times \{1\}\)

Ozsváth, Stipsicz, Szabó: associate to concordance class of a knot \(K\) a PL function

\[Y_K : [0, 1] \to \mathbb{R}\]

**Brandenbursky '11:** The following is well-defined

\[\bar{\nu} : B_n \to \text{Cont } [0, 1]\]

\[\beta \mapsto \lim_{k \to \infty} \frac{Y_{\beta^k \epsilon_k}}{k}\]

where \(\epsilon_k\) is a shortest possible word s.t. \(\beta^k \epsilon_k\) a knot
**Thm 3:**

Fix $n \geq 2$ for any $n$-braid $\beta$ we have

\[
\widehat{v}_\beta(t) = \begin{cases} 
-t \frac{wr(\beta)}{2} & \text{if } t \leq \frac{\beta}{n} \\
-t \frac{wr(\beta)}{2} + \gamma(\beta)n(t - \frac{\beta}{n}) & \text{if } \frac{\beta}{n} \leq t \leq \frac{2}{n-1}
\end{cases}
\]

Putting it all together to prove them 1

**Jones conj.**

\[
\sum_{m}^{n} \mathcal{F}_{m} \text{ braid index } \beta = 2
\]

\[
|wr(\beta) - wr(\alpha)| \leq n - m
\]

**Prop (Feller-H)** \( \beta \) concordant to \( \alpha \)

\[
|\widehat{v}_\beta(t) - \widehat{v}_\alpha(t)| \leq \epsilon \frac{n-1+m-1}{2}
\]

\[ t \in [0,1] \]

\[ \vdots \]