Contact invariant from Heegaard Floer homology

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Joint with: Földvári, Hendricks, Licata, Petkova, and Vértesi

Overview

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- a smooth, oriented nowhere integrable 2-plane field

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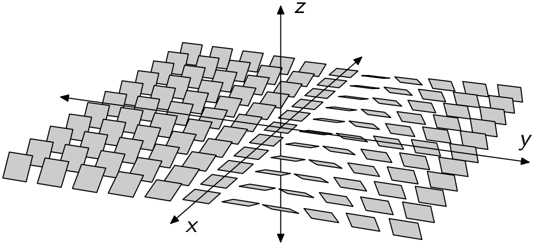
Example: $\xi = \ker(dz - ydx)$ on $\mathbb{R}^3$ (standard contact structure)



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Closed Ori. 3-manifold M $\xrightarrow{\text{Ozsváth-Szabó}}$ **Heegaard Floer homology**
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Properties:

- If ξ *overtwisted* then $c(\xi) = 0$.
- If ξ *Stein fillable* then $c(\xi) \neq 0$.

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   algebra $\mathcal{A}(\mathcal{Z})$
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$\widehat{BSD}$ and $\widehat{BSA}$

- Gluing formula that recovers SFH
- bimodule $\mathbf{W}$ s.t. $SFH(X, \Gamma) \cong H_\star(\widehat{CFA}(X, \mathcal{Z}) \boxtimes \mathbf{W}) = H_\star(\widehat{CFA}(X, \mathcal{Z})) \cdot \iota$ Cap

Main result

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- If $(M, \xi) = (-X, \xi, -\mathcal{F}) \cup_\partial (Y, \xi, \mathcal{F})$ then under the gluing formula

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Properties:

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- $(X,\xi,\mathcal{F})$: under the isom. $SFH(-X,-\Gamma) \cong H_\star(\widehat{CFA}(-X,\overline{\mathcal{Z}}))\cdot\iota$

$$[\mathbf{c_A}(X,\xi,\mathcal{F})]\cdot\iota$$

  identifies with $\mathbf{EH}(\xi)$.

Open Books

$$(M,\xi) \quad \xleftrightarrow{\text{Giroux}} \quad \left\{ \begin{array}{c} \text{Open book decomposition} \\ \\ (B,\pi) \end{array} \right\}_{/\text{stab.}} \quad \xrightarrow[\text{HKM}]{\text{OS}} \quad c(\xi)$$

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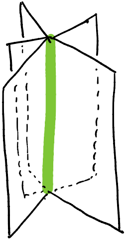

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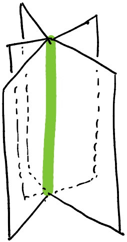

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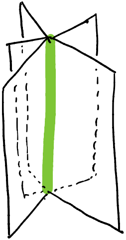
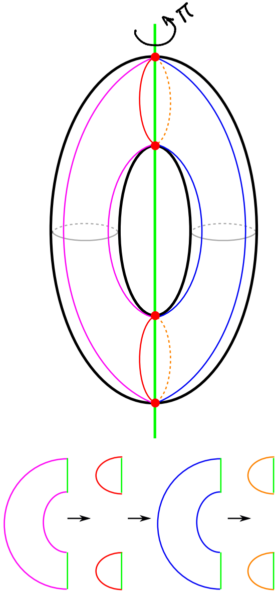
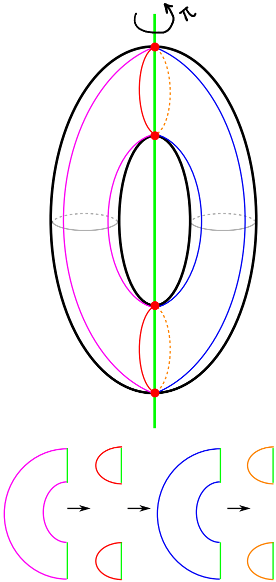


$$(X,\xi,\Gamma) \xleftrightarrow{\text{HKM}} \left\{ \text{Partial open book decomp.} \right\}_{/\text{stab.}} \xrightarrow{\text{HKM}} \mathrm{EH}(\xi)$$

Foliated open books





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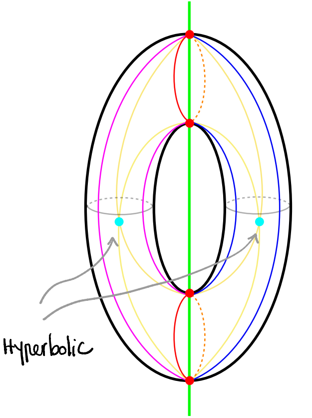

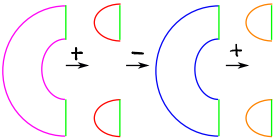

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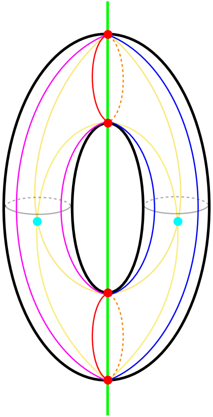

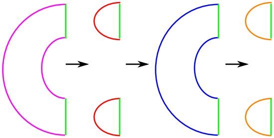

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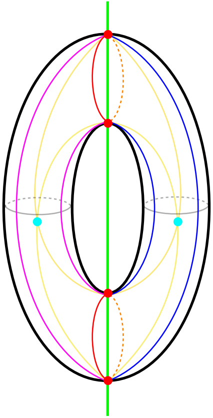

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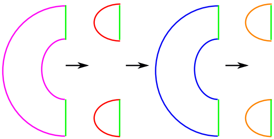
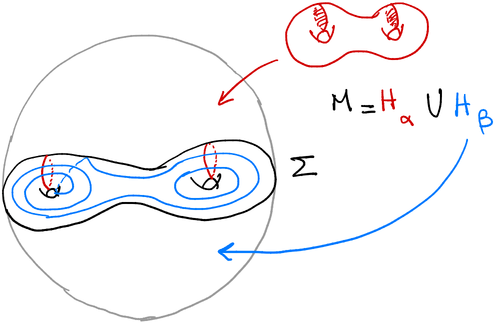


$(X, \xi, \mathcal{F}) \xleftrightarrow{\text{Licata-Vértesi}} \{\text{Fol open book}\}_{/\text{stab}}$

Heegaard Diagram for M:



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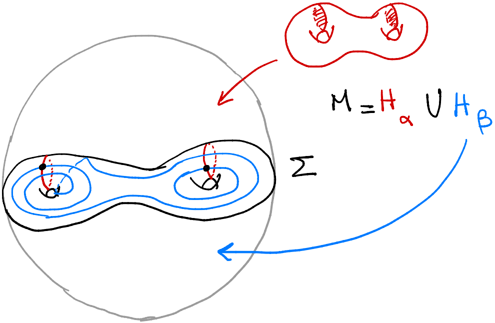
$$(\Sigma, \alpha, \beta, z)$$



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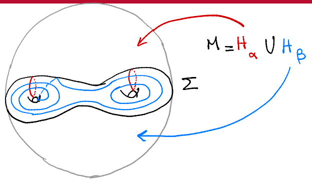
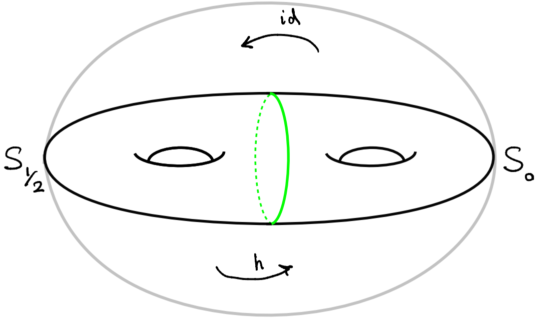


(B, π): open book decomposition for (M, ξ)



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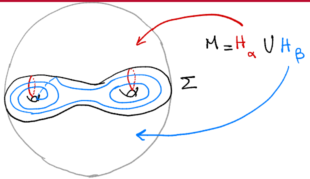

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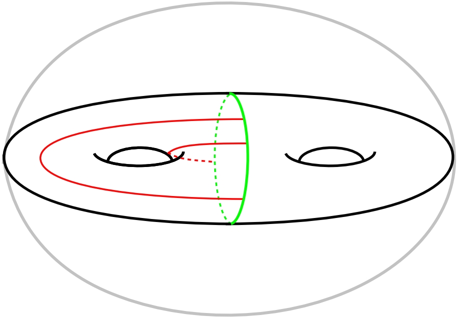

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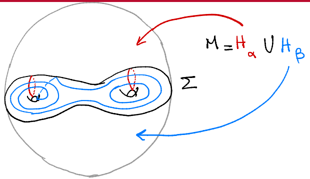

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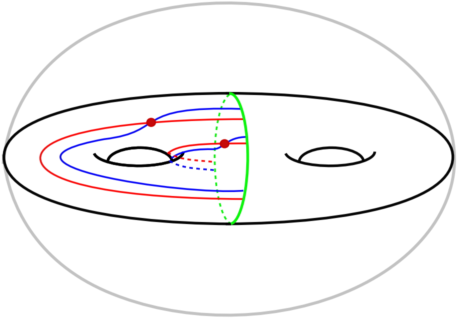

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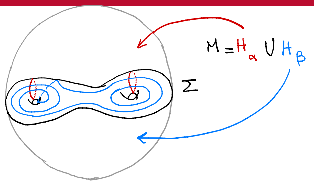

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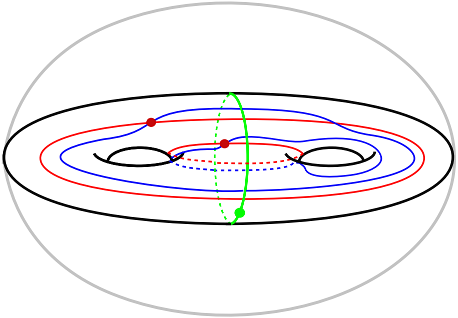

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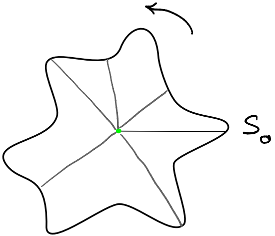
- $\beta_i = b_i \cup h(b_i)$

Definition of c_A:



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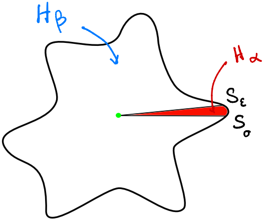

Bordered Heegaard diagram:

- $H_\alpha = \pi^{-1}[0, \epsilon]$ and $H_\beta = \pi^{-1}[\epsilon, 1]$
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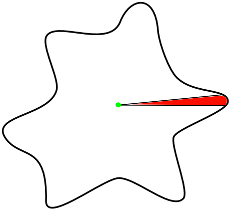

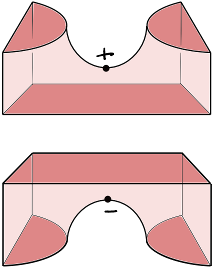

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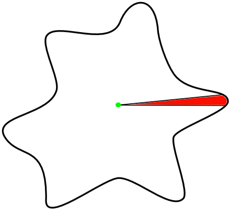

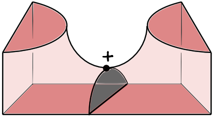

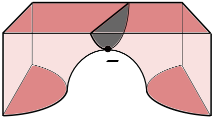

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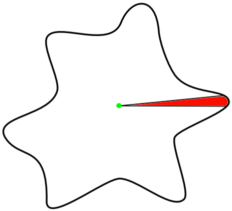

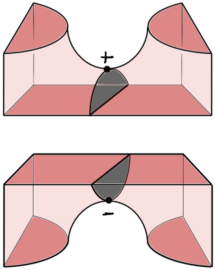

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- Any FOB can be made sorted via enough stabilizations.

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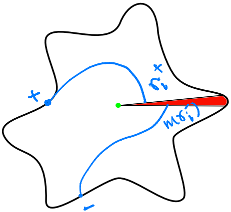

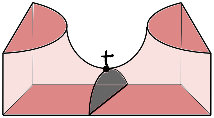

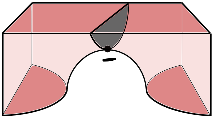

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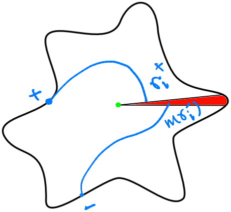

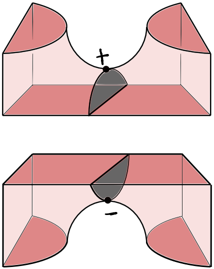

- $(B, \pi, \mathcal{F})$ is called **sorted** if for a *gradient* vector field of π critical submanifolds are disjoint in $X \setminus S_0$.
- Any FOB can be made sorted via enough stabilizations.

Bordered Heegaard diagram:

- $H_\alpha = \pi^{-1}[0, \epsilon]$ and $H_\beta = \pi^{-1}[\epsilon, 1]$
- $\Sigma = -S_0 \cup_B S_\epsilon$
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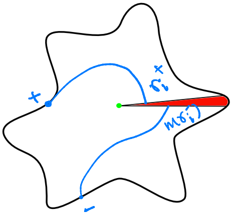

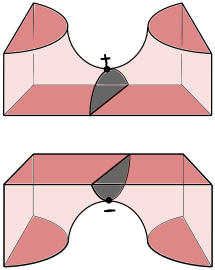

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- add basepoints

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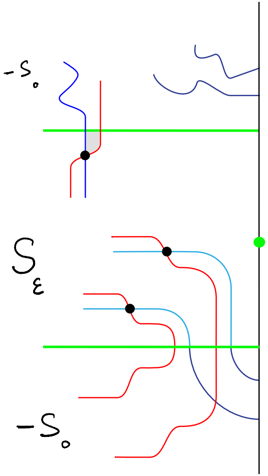

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Thank you!