Mapping class groups of connect sums of $S^2 \times S^1$

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Mapping class group

Definition

Mapping class group of closed oriented Riemannian n-manifold M^n is

$$\mathrm{Mod}(M^n) \equiv \mathrm{Diff}^+(M^n) \Big/ \mathrm{Diff}^0(M^n).$$

π_1 functor gives homomorphism

$$\mathrm{Mod}(M^n) \to \mathrm{Out}(\pi_1(M^n)).$$

Theorem (Dehn–Nielsen–Baer)

For 2-manifolds above is injective (isomorphism for $\mathrm{Mod}^{\pm}$*).*

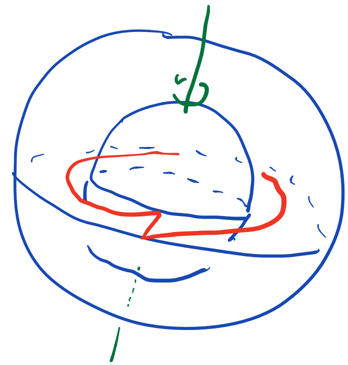
Mapping class group of 3-manifold

Question

M^3 closed oriented Riemannian 3-manifold. What is kernel

$$\mathrm{Mod}(M^3) \to \mathrm{Out}(\pi_1(M^3))?$$

Sphere twist subgroup



Definition (Sphere twist subgroup)

$S \subset M^3$ an embedded 2-sphere. **Sphere twist**

$$T_S : M^3 \to M^3$$

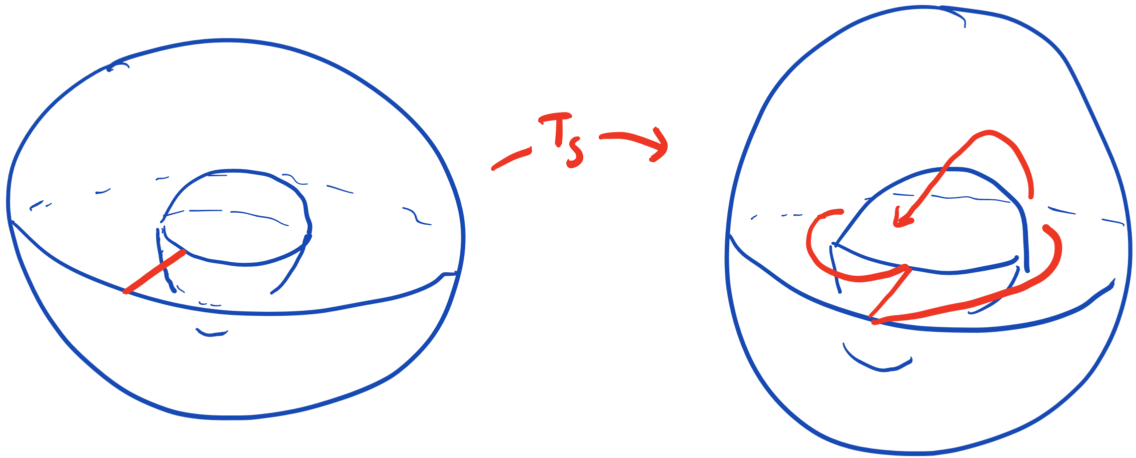
is 360° rotation in regular neighborhood of S.

$$\pi_1(\mathrm{SO}(3)) \cong \pi_1(\mathbf{R}\mathrm{P}^3) \cong \mathbf{Z}/2 \qquad \text{so } T_S^2 = 1.$$

Sphere twist subgroup of $\mathrm{Mod}(M^3)$ is subgroup

$$\mathrm{Twist}(M^3) \equiv \langle T_S \,|\, S \subset M^3 \text{ embedded sphere} \rangle$$

“Pull over the pole” idea



Sphere twist subgroup

Properties of Twist(M^3)

1. Twist(M^3) $\lhd$ Mod(M^3) since $fT_Sf^{-1} = T_{fS}$.
2. Twist(M^3) abelian since $T_S S' \simeq S'$.
3. Twist(M^3) acts trivially on Out($\pi_1(M^3)$).
4. In fact Twist(M^3) is kernel of action on Out($\pi_1(M^3)$). (See Hatcher-Wahl 2010)

pull S' over pole of S

$T_{S'} T_S T_{S'}^{-1} = T_{T_{S'}S} = T_S$

Connect sums of $S^2 \times S^1$

Set

$$M_n = \#_n S^2 \times S^1 \qquad \text{so } \pi_1(M_n) \cong F_n$$

Theorem (Laudenbach 1973)

$$\mathrm{Twist}(M_n) \cong (\mathbf{Z}/2)^n.$$

Have exact sequence Laudenbach sequence

$$1 \to \mathrm{Twist}(M_n) \to \mathrm{Mod}(M_n) \to \mathrm{Out}(F_n) \to 1$$

Nontriviality of Sphere Twists

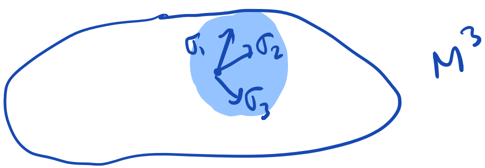
How to show T_S nontrivial in $\mathrm{Mod}(M^3)$?

1. First idea: Get action on (homotopy class of) submanifold in M^3
2. "Pull over pole" idea shows T_S acts trivially on all such homotopy classes.
3. Laudenbach uses framed cobordism and Pontryagin-Thom construction

Trivializations of TM^3

1. M^3 closed oriented 3-manifold hence **parallelizable**.
2. That means we have smooth, everywhere linearly independent sections $\sigma_1, \sigma_2, \sigma_3 : M^3 \to TM^3$ of tangent bundle.
3. That means we have global section $\sigma : M^3 \to \mathrm{Fr}(TM^3)$ of **tangent frame bundle** $(\sigma = (\sigma_1, \sigma_2, \sigma_3))$.

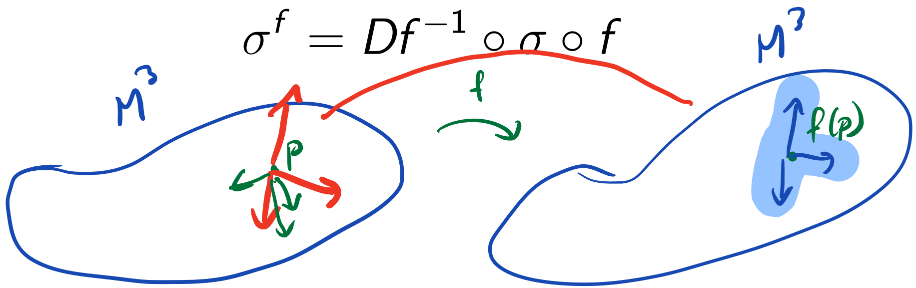
$$\mathrm{Triv}(M^3) \equiv \{\text{sections of } \mathrm{Fr}(TM^3)\}$$



Trivializations of TM^3

1. $\mathrm{Diff}^+(M)$ acts on $\mathrm{Triv}(M^3)$ via

$$\sigma^f = Df^{-1} \circ \sigma \circ f$$



2. Set $\mathrm{HTriv}(M^3) = \{\text{homotopy classes of } \mathrm{Triv}(M^3)\}$

Trivializations of TM^3

Fix **base trivialization** $\sigma_0 : M^3 \to \mathrm{Fr}(TM^3)$ and let $[\sigma_0]$ be its homotopy class.

Theorem (Brendle-B-Putman 2020)

1. $\mathrm{Twist}(M_n) \cong \mathrm{H}^1(M_n; \mathbf{Z}/2)$ *as* $\mathrm{Mod}(M_n)$*-modules.*
2. $(\mathrm{Mod}(M_n))_{[\sigma_0]} \cong \mathrm{Out}(F_n)$.
3. $\mathrm{Mod}(M_n) = \mathrm{Twist}(M_n) \rtimes (\mathrm{Mod}(M_n))_{[\sigma_0]}$

(Laudenbach sequence splits)

Nontriviality of Sphere Twists

$$M_n = \#_n S^2 \times S^1$$

- We streamline Laudenbach proof that "core sphere twists" generate Twist(M_n).
- Today I focus on our nontriviality of Twist(M_n) proof.

Maps to structure group

- Given trivializations

$$\sigma, \tau \in \mathsf{Triv}(M^3)$$

- For $p \in M^3$ have matrix $\phi_{\sigma,\tau}(p) \in \mathsf{GL}_3(\mathbf{R})$ taking basis $\sigma(p)$ to basis $\tau(p)$.

- Get smooth

$$\phi_{\sigma,\tau} : M^3 \longrightarrow \mathsf{GL}_3(\mathbf{R})$$

Derivative crossed homomorphism

- Let

$$C(M^3, \mathrm{GL}_3^+(\mathbf{R}^3)) = \{\text{smooth } \phi : M^3 \to \mathrm{GL}_3^+(\mathbf{R})\}$$

- $C(M^3, \mathrm{GL}_3^+(\mathbf{R}^3))$ has $\mathrm{Diff}^+(M^3)$-action given by

$$\phi^f = \phi \circ f$$

- Define

$$\mathscr{D} : \mathrm{Diff}^+(M^3) \to C(M^3, \mathrm{GL}_3^+(\mathbf{R}^3))$$

by

$$\mathscr{D}(f) = \phi_{\sigma_0^f, \sigma_0}$$

base trivialization σ_0

Derivative crossed homomorphism

- $$\mathscr{D} : \mathrm{Diff}^+(M^3) \to C(M^3, \mathrm{GL}_3^+(\mathbf{R}^3))$$

  is a **crossed homomorphism** meaning

  action of Diff$^+$ on $C(M^3, GL)$

  $$\mathscr{D}(f_1 f_2) = \mathscr{D}(f_1)^{f_2} \mathscr{D}(f_2)$$

- Taking homotopy classes get crossed homomorphism

  $$\mathscr{D} : \mathrm{Mod}(M^3) \to [M^3, \mathrm{GL}_3^+(\mathbf{R}^3)]$$

Twisted crossed homomorphism

$\pi_1(GL_3^+) \simeq SO(3) \simeq \mathbb{RP}^3$

- π_1 functor gives homomorphism

$$\pi_1 : [M^3, \mathrm{GL}_3^+(\mathbf{R}^3)] \to \mathrm{Hom}(\pi_1(M^3), \mathbf{Z}/2) = \mathrm{H}^1(M^3; \mathbf{Z}/2)$$

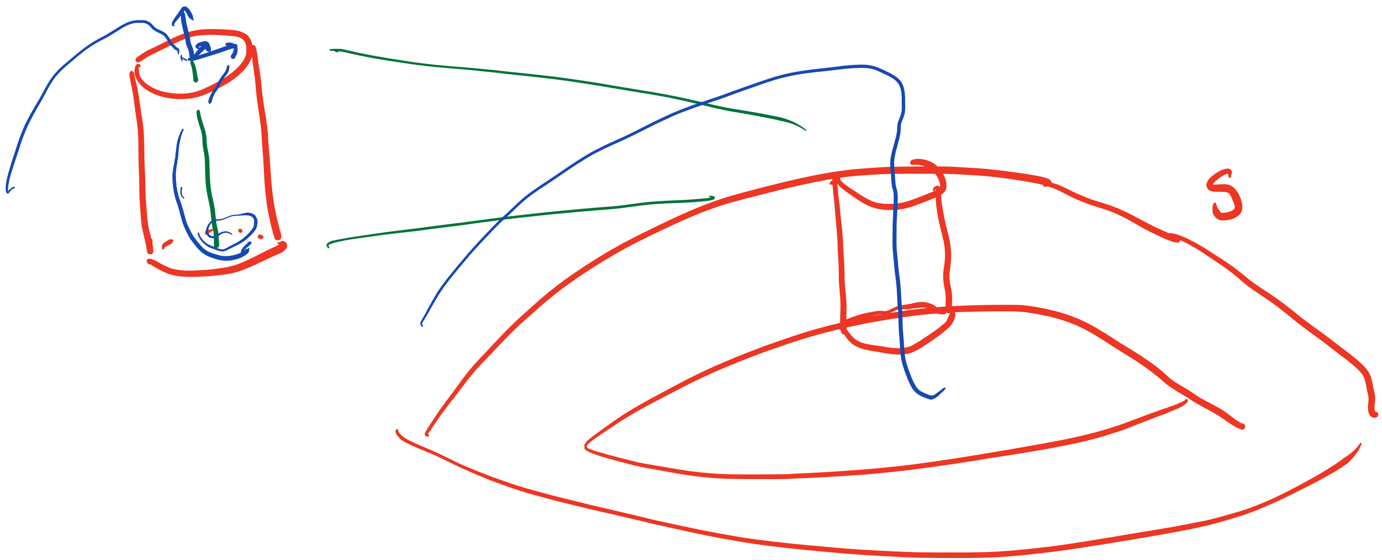
- Composition $\mathscr{T} = \pi_1 \circ \mathscr{D}$ is **twisted crossed homomorphism**

$$\mathscr{T} : \mathrm{Mod}(M^3) \to \mathrm{H}^1(M^3; \mathbf{Z}/2)$$

Image of sphere twist under $\mathscr{T}$

Lemma

$S \subset M^3$ embedded sphere. Then $\mathscr{T}(T_S) \in \mathrm{H}^1(M^3; \mathbf{Z}/2)$ is Poincaré dual of $[S] \in \mathrm{H}_2(M^3; \mathbf{Z}/2)$



Laudenbach Sequence Splits

Corollary

Conjugation action of $\mathrm{Mod}(M_n)$ *on* $\mathrm{Twist}(M_n)$ *is same as action on* $\mathrm{H}^1(M^3; \mathbf{Z}/2)$*.*

Lemma

$$1 \to A \to G \to Q \to 1$$

splits if and only if there is crossed homomorphism $G \to A$*.*