Pure Braids and Link Concordance

Miriam Kuzbary

Georgia Tech

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Talk Outline:

I. Background
II. String Link Concordance Group
III. Milnor's Invariants

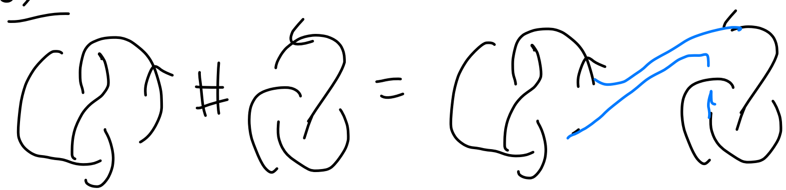
I. Background

In this talk: everything is smooth and oriented

Recall: There is a binary op on knots $\subseteq S^3$

$$K \# J := (S^3, K) \# (S^3, J)$$

Ex



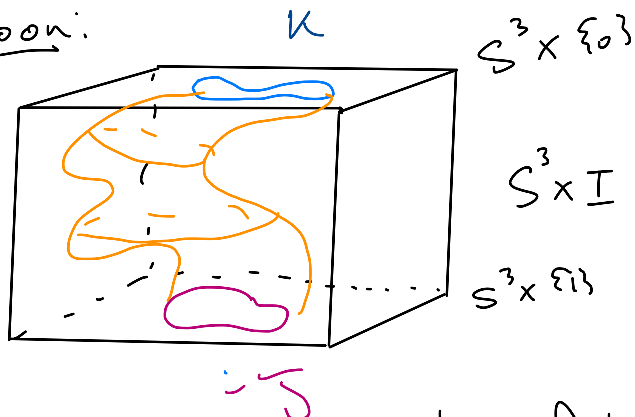
Fact: $(\{K \subseteq S^3\}, \#)$ is a monoid, not a group!

Exercise: Inverses don't exist b/c genus is additive under $\#$

Q: Where would we get inverses from, then?

Def: Knots $K, J \subseteq S^3$ are concordant if there is a smooth, properly embedded annulus $C \subseteq S^3 \times I$ with $\partial C = K \sqcup -J$

Cartoon:



Def A knot $K \subseteq S^3$ is slice if there is a smooth, properly embedded disk $\Delta \subseteq B^4$ with $\partial \Delta = K$

The Knot Concordance Group

Theorem (Fox-Milnor '66)

The set of knots $K \subset S^3$ with the operation connected sum forms a monoid, this monoid modulo concordance forms the knot concordance group $\mathcal{C}$.

$\mathcal{C}$ has the following properties:

- [Fox-Milnor '66] $\mathcal{C}$ is abelian,
- [Fox-Milnor '66] $\mathcal{C}$ has elements of finite order (namely, order 2),
- [J. Levine '69] $\mathcal{C}$ surjects onto $\mathcal{A} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$.
  - $\mathcal{A}$ is known as the algebraic concordance group.
  - [Casson-Gordon '75] The kernel of this map is nontrivial.
  - [Jiang '81] The kernel of this map contains a $\mathbb{Z}^\infty$ subgroup.
  - [Livingston-Naik '99] Many of the preimages of order 4 elements $\mathcal{A}$ are infinite order in $\mathcal{C}$.
- [Cochran-Orr-Teichner '03] There is a geometric filtration of $\mathcal{C}$ whose successive quotients are non-trivial.
  - [Harvey '08] The successive quotients of the COT filtration have infinite rank.

Q: What about links?

There is a notion of link concordance

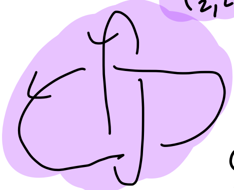
- **Strong:** n-cpt links L_1, L_2 are strongly concordant if their components are concordant by disjoint annuli
- **Weak:** L_1, L_2 are weakly concordant if $\exists$ smooth, properly embedded genus 0 sfc $\Sigma \subseteq S^3 \times I$ with $\partial \Sigma = L_1 \sqcup -L_2$

There's a bigger problem here...

Connected sum isn't well-defined for links, even oriented, ordered ones.

Pf Consider

$T_{2,2}$

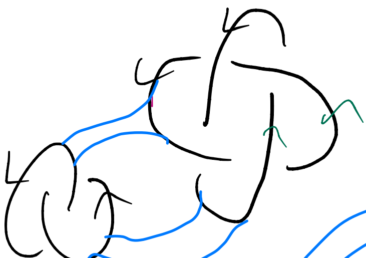


"#"

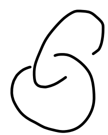
H^+ =

could take oriented, ordered "connected sum" in a few ways, such as:

(1)



$\sim$ isotopic

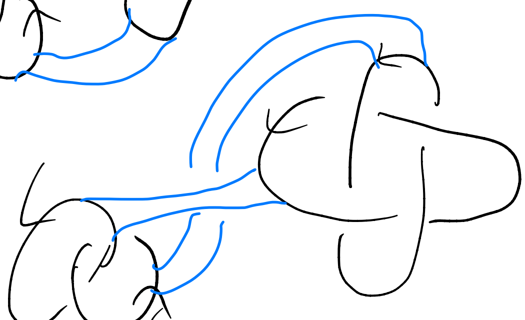


H^-

or

(2)

$L =$



If L and H^- were concordant (and # was well defined on concordance classes) then

L "#" H^+

H^- "#" H^+ $\sim_{conc}$

"

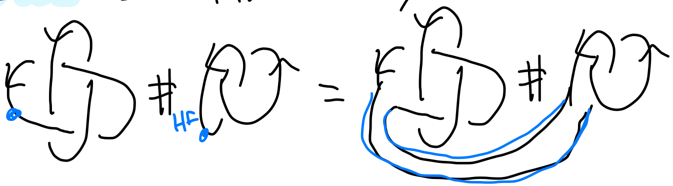
$= \bigcirc \; \bigcirc$

L "#" H^+ =

isotopic to 9^2_9 (L9a35) which has $\sigma = 1$

not even weakly slice

Aside: In HF world, often define



Notice: n-cpt $\#_{HF}$ n-cpt = 2n-1 cpt link

If you want a group of n-component links, this won't do it for you [See Donald-Owens 2012]

II. The String Link Concordance Group

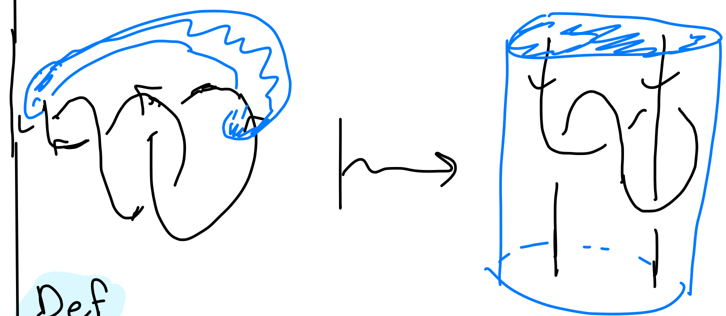
Idea: Base a link with a disk and "connect sum" along the disk.

Ex

ordered, oriented link $L \subseteq S^3$

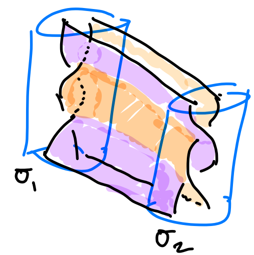
ordered, n-cpt string link $\sigma \subseteq D^2 \times I$

(!) Representatives are not unique



Def

Two n-component string links σ_1 and σ_2 are concordant if there is a smooth embedding $H : \bigsqcup_n (I \times I) \to B^3 \times I$ which is transverse to the boundary such that



$$H|_{(\bigsqcup_n I \times \{0\})} = \sigma_1$$

$$H|_{(\bigsqcup_n I \times \{1\})} = \sigma_2$$

$$H|_{(\bigsqcup_n \partial I \times I)} = j_0 \times id_I \text{ with } j_0 : \bigsqcup_n \partial I \to S^2.$$

Theorem (Habegger-Lin '98)

An n-component string link $\sigma \subset D^2 \times I$ is concordant to the trivial string link if and only if its closure $\widehat{\sigma} \subset S^3$ is strongly concordant to the n-component unlink (i.e. is strongly slice).

The String Link Concordance Group

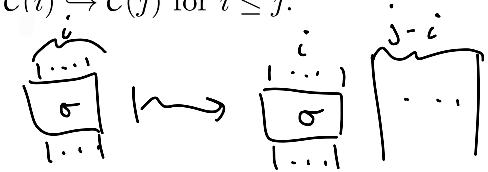
Definition (Le Dimet '88)

$\mathcal{C}(n) = (\frac{\text{n-component string links}}{\text{string link concordance}}, \text{stacking})$ is a group.

Notice: • {pure braids} $\subsetneq$ {string links} $\subsetneq$ {tangles}

• $\mathcal{C}(1) \cong \mathcal{C}$ ← Knot concordance group

- $\mathcal{C}(i) \hookrightarrow \mathcal{C}(j)$ for $i \leq j$.



- [Le Dimet '88] The pure braid group $\mathcal{P}(n) \hookrightarrow \mathcal{C}(n)$.
- [Le Dimet '88, De Campos '95 for $n = 2$] $\mathcal{C}(n)$ is non-abelian;
- [Cochran-Orr-Teichner '03] There is a geometric filtration $\mathcal{F}_{(m)}$ of $\mathcal{C}(n)$ whose successive quotients are non-trivial.
- [Harvey '08] The abelianization of successive quotients of $\mathcal{F}_{(m)}$ has infinite rank.
- [Cha '08] There are infinitely many order 2 elements in $\mathcal{F}_{(m)}$ that are not in $\mathcal{F}_{(m+2)}$.

Q: How much of the non-abelian structure of $\mathcal{C}(n)$ is inherited from $P(n)$?

Theorem (Kirk-Livingston-Wang '98)

The pure braid group $\mathcal{P}(n)$ is not a normal subgroup of $\mathcal{C}(n)$ for $n \geq 2$.

Thm (K. 2020)

$\mathcal{C}(n) / Ncl(P(n))$ is non-abelian for all n.

Morally, this a complementary result to the celebrated classification of links up to link homotopy by Habegger-Lin '90, where they show

$$\mathcal{C}(n)/\text{link htpy} \twoheadrightarrow P(n)/\text{link htpy}$$

III. Milnor's Invariants

The main tool in the proof of this theorem is a specific subset of Milnor's invariants which are trivial for elements of $Ncl(P(n))$

Recall

If L is an n-component link with L_i the 0-framed longitude of the i^{th} component,

$$[L_i] = \sum_{i=1}^{n} \mathrm{lk}(L_i, L_j)\ x_j \in \mathrm{H}_1(S^3 \setminus \nu(L))$$

where x_i generate $\mathrm{H}_1(S^3 \setminus \nu(L))$.

Let $G = \pi_1(S^3 \setminus \nu(L), *)$.

Notice that

$$H_1(S^3 \setminus \nu(L)) \cong \frac{G}{[G,G]} \cong \mathbb{Z}^n$$

Q: what happens if you look at image of the i^{th} longitude in another quotient of G?

Recall: The lower central series of G is

$G_1 = G, \quad G_{n+1} = [G, G_n]$

Concordance Data in the Lower Central Series

Theorem (Casson '75)

If L_1 and L_2 are concordant links in S^3 with groups $G = \pi_1(S^3 \setminus \nu(L_1), *)$ and $H = \pi_1(S^3 \setminus \nu(L_2), *)$, then G/G_q and H/H_q are isomorphic for all q.

Corollary (Casson '75)

If L_1 and L_2 are concordant links in S^3 whose 0-framed longitudes are $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$, then $y_i \in G_q$ if and only if $z_i \in H_q$.

Rough definition (Milnor '54)

The Milnor invariants of an n-component link $L \subset S^3$ with link group G are a set of integer-valued link (and string link) concordance invariants

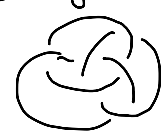
$$\overline{\mu}_L(I) \in \mathbb{Z}$$

with $I = (i_1 \ldots i_k)$ and $i_j \in \{1, \ldots, n\}$. They detect how deep the i_k^{th} longitude of L is in G_q.

Idea: $\overline{\mu}_L(I)$ are higher order linking #'s

Ex

- $\overline{\mu}_L(ij) = lk(L_i, L_j)$
- $\overline{\mu}_L(ijk) =$ triple linking #

BR =  has $\overline{\mu}_{BR}(123) = 1$

Unfortunate Fact

$\overline{\mu}_L(I)$ are generally defined only up to the gcd of a subset of $\overline{\mu}_L(I)$ of lower order.

Fortunate Fact

If we are computing Milnor's invariants for string links instead of links, they are well defined without quotienting by anything! (see Levine, Habegger-Lin)

Properties of Milnor's Invariants

- [Casson '75] $\overline{\mu}_L(I)$ are link concordance invariants.
- [Turaev '79, Porter '80] $\overline{\mu}_L(I)$ can be computed by evaluating Massey products on $\partial(S^3 \setminus \nu(L))$.
- [Cochran '90] The first non-vanishing $\overline{\mu}_L(I)$ can be computed using iterated intersections of surfaces.
- [Conant-Schneiderman-Teichner '15] The first non-vanishing $\overline{\mu}_L(I)$ can be computed using intersection trees of twisted Whitney towers.
- [Gorsky-Liu-Moore '20] For 2-component links with linking number 0, can compute $\overline{\mu}_L(1122)$ from link Floer homology.
- [Gorsky-Lidman-Liu-Moore '20] For 3-component links, can compute $\overline{\mu}_L(123)$ from link Floer homology.

Further Questions

- What is the abelianization of $\mathcal{C}(n) / Ncl(P(n))$?
- Is $\mathcal{C}(n) / Ncl(P(n))$ solvable?
- Does $\mathcal{C}(n) / Ncl(P(n))$ have finite order elements? Note that $P(n)$ is torsion free
  - What is the structure of $Ncl(P(n))$?
  - Are there boundary links in it?
- (Harvey-Park-Ray) How does the pure braid group interact with the (n)-solvable filtration?

Thank you!!!