The cosmetic surgery conjecture for pretzel knots

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Surgeries

Suppose that $Y = Y^3$ is a closed, oriented three-manifold, $K \subset Y$ a framed knot and $r \in \mathbb{Q} \cup \{\infty\}$ a surgery coefficient

- Dehn surgery associates to this data a new three-manifold $Y_r(K) = (Y \setminus \nu(K)) \cup_\varphi S^1 \times D^2$. (The identification φ is determined by r.)
- The notion naturally extends to framed links.

Theorem (Lickorish, Wallace)

For any Y there is a link $L \subset S^3$ (each knot equipped with the Seifert framing) and $R = (r_1, \ldots, r_n) \in \mathbb{Q}^n$ so that $S^3_R(L)$ is orientation preserving diffeomorphic to Y.

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Surgeries

- The link is not unique — different choices can be connected by Kirby moves. Not even if we assume the the link is a knot: 5-surgery along the RHT is the same as (-5)-surgery along the unknot (giving the lens space $L(5,1)$)
- Sometimes the knot and the coefficient is determined by the three-manifold: the Poincaré homology sphere $\Sigma(2,3,5)$ can be only surgered along the (LH) trefoil with $r=-1$. Similarly, $S^1 \times S^2$ is surgery only along the unknot with framing 0.
- the projective space $\mathbb{RP}^3$ can be given by surgery only along the unknot (framing: ± 2),

The (purely) cosmetic surgery conjecture, PCSC

"For a fixed knot the result determines the surgery coefficient."

Conjecture (Gordon, 1990)

Suppose that $K \subset S^3$ is a non-trivial knot. Suppose that for $r, s \in \mathbb{Q}$ we have that $S^3_r(K)$ and $S^3_s(K)$ are orientation preserving diffeomorphic three-manifolds. Then $r = s$.

If we drop 'orientation preserving', the situation is very different: we always have that $S^3_r(K)$ and $S^3_{-r}(m(K))$ for the mirror $m(K)$ are (orientation-reversing) diffeomorphic. Hence if K is amphichiral, r and $-r$ give the same three-manifold; and there are further examples.

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Results

Theorem (Wang)

If $g(K) = 1$, then K satisfies the purely cosmetic surgery conjecture.

Theorem (Ni-Wu)

Suppose that for a nontrivial knot K we have that $S^3_r(K) \cong S^3_s(K)$ with $r \neq s$. Then $r = -s$.

So we need to compare $S^3_r(K)$ and $S^3_{-r}(K)$.

Results

The PCS Conjecture holds for:

- torus knots
- nontrivial connected sums, and cable knots (R. Tao)
- 3-braid knots (Varvarezos)
- two-bridge knots and alternating fibered knots (Ichihara-Jong-Mattman-Saito)
- Conway and Kinoshita-Terasaka knot families (Bohnke-Gillis-Liu-Xue)
- knots up to 16 crossings (Hanselman)
- Today: Pretzel knots (S-Szabó)

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Knot Floer homology

Knot Floer homology: it associates a finite dimensional bigraded vector space (over $\mathbb{F} = \{0, 1\}$)

$$\widehat{\mathrm{HFK}}(K) = \sum_{M,A} \widehat{\mathrm{HFK}}_M(K, A)$$

to a knot. It determines: Seifert genus and fiberedness.

There is a surgery formula, providing relation between $\widehat{\mathrm{HFK}}(K)$ and $\widehat{\mathrm{HF}}(S^3_r(K))$.

Thickness of knots

Suppose that $V = \bigoplus_{a\in\mathbb{R}} V_a$ is a finite dimensional graded vector space, V_a is the subspace of homogeneous elements of grading a.

Definition

The thickness $th(V)$ of the vector space V is the largest possible difference of degrees, i.e.

$$th(V) = \max\{a \mid V_a \neq 0\} - \min\{a \mid V_a \neq 0\}.$$

(For example, the thickness of $H_*(M^n; \mathbb{Z}/2\mathbb{Z})$ for an n-dimensional closed manifold is n.)

Collapse the two gradings of $\widehat{\mathrm{HFK}}(K)$ to $\delta = A - M$; the thickness of the resulting graded vector space $\widehat{\mathrm{HFK}}^{\delta}(K)$ is, by definition, the thickness $th(K)$ of K.

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Main technical result

Theorem (Hanselman)

Suppose that the nontrivial knot K admits $r \neq s$ with $S^3_r(K) \cong S^3_s(K)$. Then, $\{r, s\}$ is either $\{\pm 2\}$, or $\{\pm \frac{1}{q}\}$ with $q \in \mathbb{N}$ determined by $\widehat{\mathrm{HFK}}(K)$, and

- *if $\{r, s\} = \{\pm 2\}$ then $g(K) = 2$;*
- *$\{r, s\} = \{\pm \frac{1}{q}\}$ for some $q \in \mathbb{N}$ then*

$$q \leq \frac{th(K) + 2g(K)}{2g(K)(g(K) - 1)},$$

where $th(K)$ is the thickness of K.

In particular, if $g(K) > 2$ and $th(K) \leq 5$, then K satisfies the purely cosmetic surgery conjecture (PCSC).

Thickness

Definition

Suppose that D is a diagram of a knot $K \subset S^3$. A domain d is good if every edge on its boundary connects an over- and an under-crossing; otherwise d is bad. Let $B(D)$ denote the number of bad domains.

The knot invariant

$$\beta(K) = \min\{B(D) \mid D \text{ is a diagram of } K\}$$

measures how far K is from being alternating.

A bound on β

Suppose that K is non-alternating (that is, $\beta(K) > 0$). Then

Theorem

- *$th(K) \leq \frac{1}{2}\beta(K) - 1$.*
- *If K is a pretzel knot or a Montesinos knot, then $th(K) \leq 1$.*

- Same result can be shown using the 'Turaev genus', another measure of how non-alternating K is.
- Combining with Zibrowius' theorem – implying that $th(K)$ is mutation invariant – one can get bounds in other cases.

Further tools for PCSC

Theorem (Boyer-Lines)

Suppose that the knot $K \subset S^3$ has Alexander-Conway polynomial $\nabla_K(z) = \sum_{i=0}^{d} a_{2i}(K) z^{2i}$ with $a_2(K) \neq 0$. Then K satisfies the PCSC.

Recall that ∇_K is defined by the skein relation

$$\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z), \qquad \nabla_U = 1$$

It satisfies the identity $\nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_K(t)$ (Δ_K: symmetrized Alexander polynomial).

Idea: connect the Casson-Walker invariant of the surgery with $a_2(K)(= \frac{1}{2}\Delta_K''(1))$.

A further invariant: $\lambda_2(Y)$ of a rational homology sphere Y is a generalization of the Casson-Walker invariant $\lambda = \lambda_1$. Admits a surgery formula, involving the knot invariants $a_2(K)$ and

$$w_3(K) = \frac{1}{72} V_K'''(1) + \frac{1}{24} V_K''(1),$$

where V_K is the (normalized) Jones polynomial of K:

Theorem (Lescop)

$$\lambda_2(S^3_{\frac{p}{q}}(K)) = \lambda_2''(K) \cdot (\frac{q}{p})^2 + w_3(K)\frac{q}{p} + a_2(K)c(p,q) + \lambda_2(L(p,q))$$

Pretzel knots

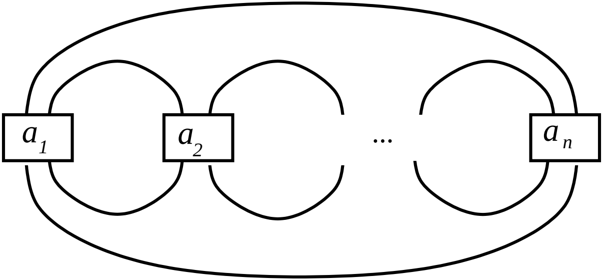

Figure: **The pretzel knot $P(a_1, \ldots, a_n)$.** The box with a_i in it means $|a_i|$ half twists (to the right if $a_i > 0$ and to the left if $a_i < 0$). We have a knot if a_1 is even and all others are odd, or all are odd and n is odd.

Pretzel knots

Theorem (S-Szabó)

Pretzel knots satisfy the PCSC.

Suppose $P = P(b_1, \ldots, b_n)$ is a pretzel knot. If $g(P) \neq 2$, then Hanselman's corollary ("$g > 2$, $th \leq 5$ implies PCSC") shows that PCSC holds.

If b_1 is even, then there are a few families of knots with $g(K) = 2$, and they can be easily handled by $a_2(P) \neq 0$. If all b_i are odd, then $g(P) = \frac{n-1}{2}$, so we need to focus on five-strand pretzels.

Five-strand pretzel knots

For a five-strand pretzel knot $P = P(b_1, \ldots, b_5)$ with $b_i = 2k_i + 1$ odd, and with s_k the k^{th} symmetric polynomial in $\{k_i\}_{i=1}^5$, we have

$$a_2(P) = s_2 + 2s_1 + 3,$$

$$w_3(P) = \frac{1}{2}(5 + 3s_1 + s_1^2 + s_2 + \frac{1}{2}(s_3 + s_1 s_2)).$$

Simple argument shows that

Proposition

The quantities $a_2(P)$ and $w_3(P)$ cannot be zero at the same time.

Idea: If both are zero, then $s_2 = -2s_1 - 3$ and $s_3 = s_1 + 2$, the first is a degree-2, the second is a degree-3 equation, so we do not expect them to be satisfied at the same time.

Suppose that D is a non-alternating diagram; we want to show that

$$th(K) \leq \frac{1}{2}B(D) - 1.$$

Recall that $th(K) = th(\widehat{\mathrm{HFK}}^{\delta}(K))$, and $\widehat{\mathrm{HFK}}^{\delta}(K)$ is the homology of a chain complex $(C_{D,p}, \partial)$ (associated to a diagram D with a marked point p), generated by Kauffman states.

(Recall: a Kauffman state of (D, p) is a bijection between crossings and domains not touching p, such that the domain associated to a crossing has the crossing in its closure.)

As $th(H(V, \partial)) \leq th(V)$ for any chain complex (V, ∂), if $C_{D,p}$ is the δ-graded vector space spanned by the Kauffman states, then it is sufficient to show that $th(C_{D,p}) \leq \frac{1}{2}B(D) - 1$.

Equip each Kauffman state by the gradings A, M (and $\delta = A - M$) as instructed by

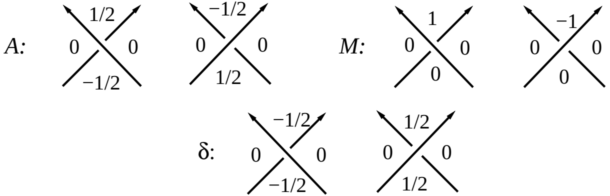

Figure: The local contributions to $M(\kappa), A(\kappa)$ and $\delta(\kappa)$.

The proof of the inequality

The δ-grading at a positive crossing is either 0 or $-\frac{1}{2}$, at a negative one either 0 or $\frac{1}{2}$. So we can express the δ-grading of a Kauffman state κ as the sum

$$-\frac{1}{4}\mathrm{wr}(D) + \sum_{c \in Cr} f(\kappa(c)),$$

where wr is the writhe of the diagram, and f is a function on the Kauffman corners, which is either $\frac{1}{4}$ or $-\frac{1}{4}$ (depending on the chosen quadrant at the crossing c).

Main observation: For a good domain each corner in the domain gives the same f-value, hence for different Kauffman states the contributions from this particular domain are the same. For a bad domain the maximal difference for two Kauffman states on a bad domain is $\frac{1}{2}$. We gain the -1 from putting p to the boundary of bad domains.

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Thank you!