Ribbon cobordisms between lens spaces

Marius Huber

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Rational homology cobordisms

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Definition

Let Y_1, Y_2 be oriented rational homology 3-spheres. A *rational homology cobordism* from Y_1 to Y_2 is a 4-manifold W such that

- $\partial W = -Y_1 \amalg Y_2$, and
- the inclusion maps $\iota_i \colon Y_i \to W$ induce isomorphisms $(\iota_i)_* \colon H_*(Y_i;\mathbb{Q}) \to H_*(W;\mathbb{Q})$, $i = 1, 2$.

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- Q. When does there exist a rational homology cobordism from one lens space to another lens space?
- Lisca (2007) completely answered this question.

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- By a standard argument involving Donaldson's Diagonalization Theorem, we obtain an embedding

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- Proof follows from a combinatorial analysis of such embeddings.

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Lemma (H. 2020)

Let L_1, L_2 be lens spaces such that $L_1 \leq L_2$, and let X_i be the canonical negative definite plumbing bounded by L_i, $i = 1, 2$. Then there exists an isometric embedding

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The proof relies on the argument from before, together with some elementary algebraic topology.

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Theorem (H. 2020)

Suppose that $L(p_1, q_1) \leq L(p_2, q_2)$, *where* $p_1 \neq p_2$. *Then, up to orientation reversal, we must have that*

$$L(p_1, q_1) \cong L(n, 1), \text{ for some } n \geq 1.$$

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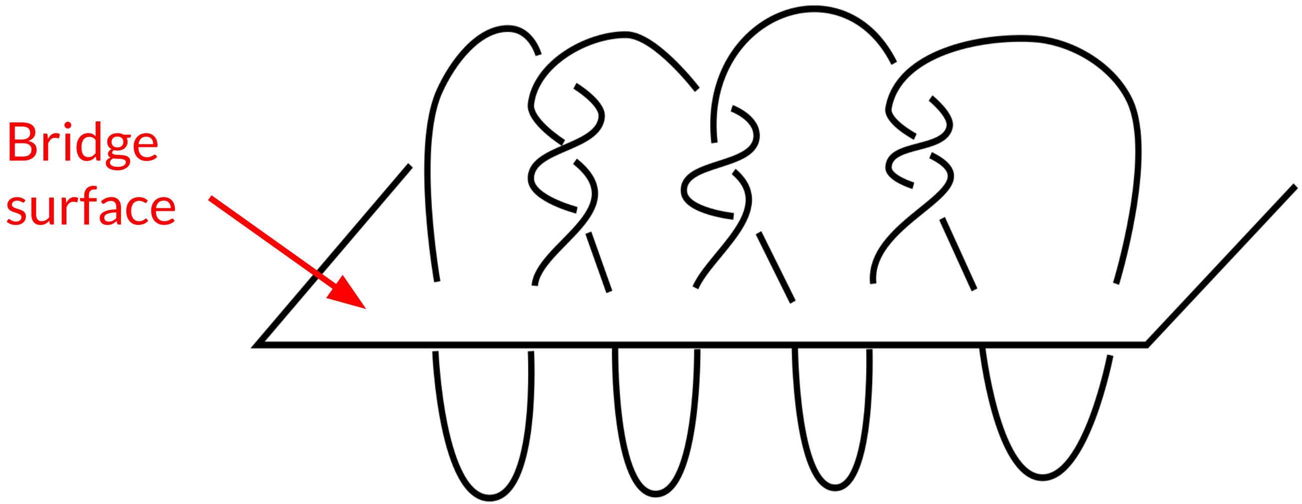
Conversely, if $L(p_2, q_2) \sim L(n, 1)$*, for some* $n \geq 1$*, then*

$$L(n, 1) \leq L(p_2, q_2).$$

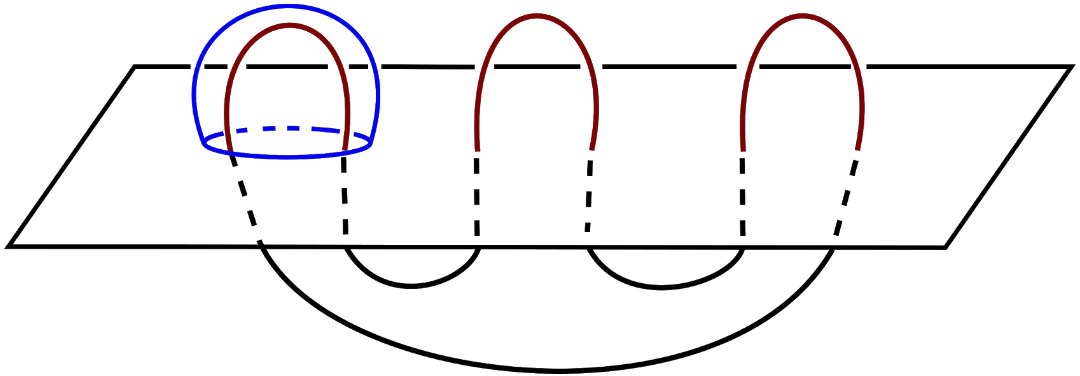
Bridge surfaces with non-contractible disk complexes

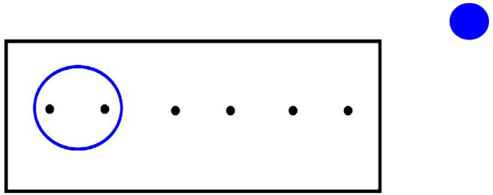
Puttipong Pongtanapaisan (University of Iowa)
Joint with Daniel Rodman

A knot can be put in ***bridge position*** so that maxima lie above minima.



A loop on the bridge surface may bound a disk on either side.





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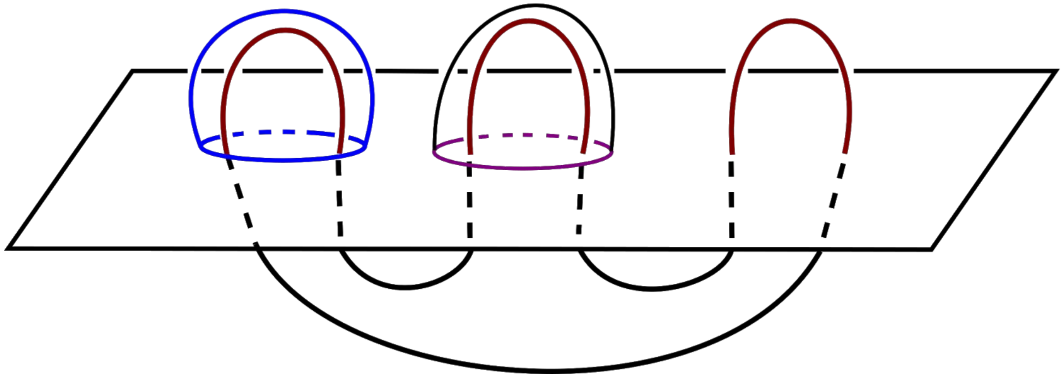

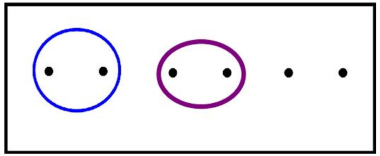

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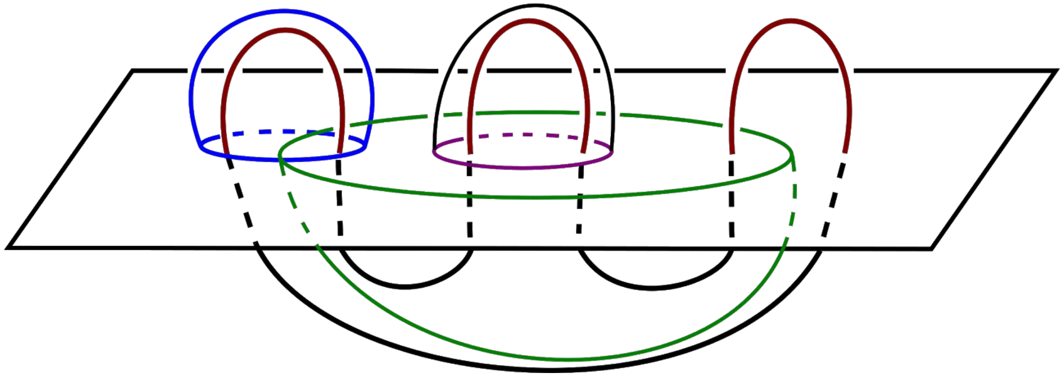

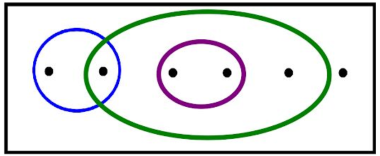

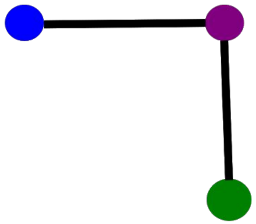

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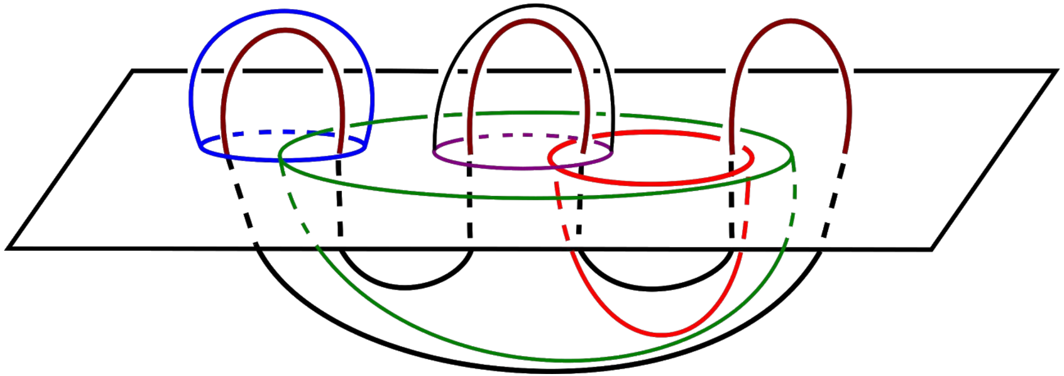
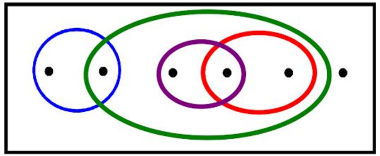
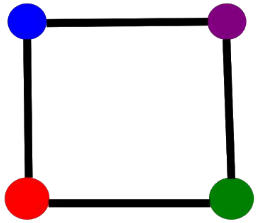




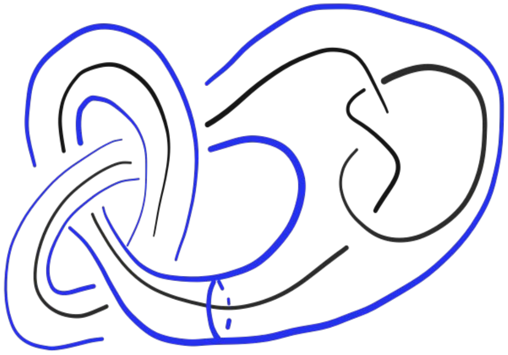


Disk complex

Useful Fact:

An incompressible surface and another incompressible surface can be isotoped to intersect only in curves that are essential in both surfaces.

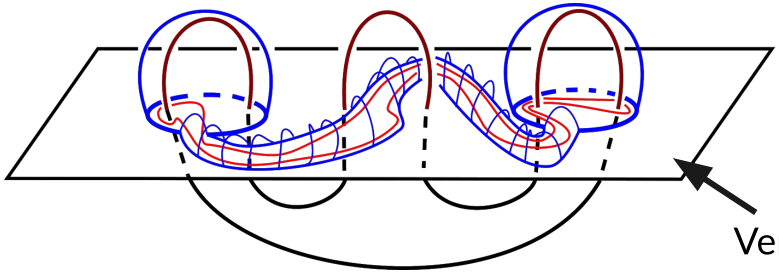
Incompressible surface



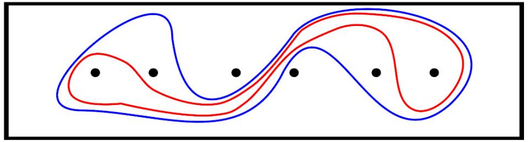
Even More Useful Fact (Bachman):

A surface whose disk complex is not contractible

~~An incompressible surface~~ and another incompressible surface can be isotoped to intersect only in curves that are essential in both surfaces.

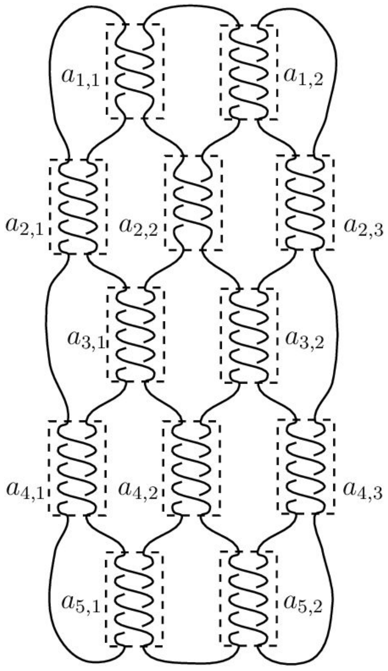


Very compressible

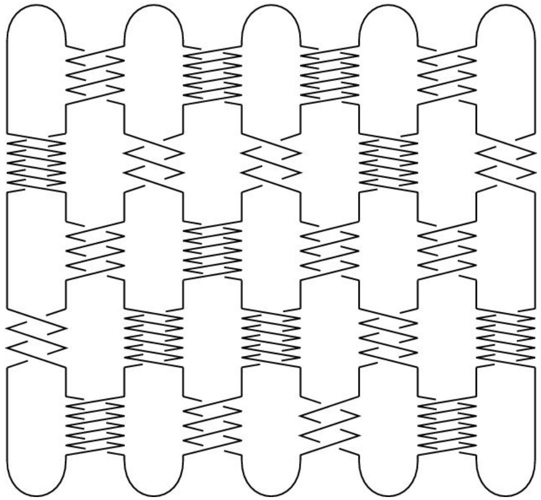


J. Johnson and Y. Moriah, ‘ Bridge distance and plat projections’, *Algebr. Geom. Topol.* 16 (2016) 3361– 3384. MR3584261.

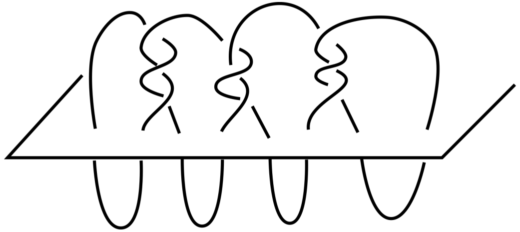
Pongtanapaisan, P., Rodman, D. Critical bridge spheres for links with arbitrarily many bridges. *Rev Mat Complut* (2020)



(Johnson-Moriah) disconnected disk complex

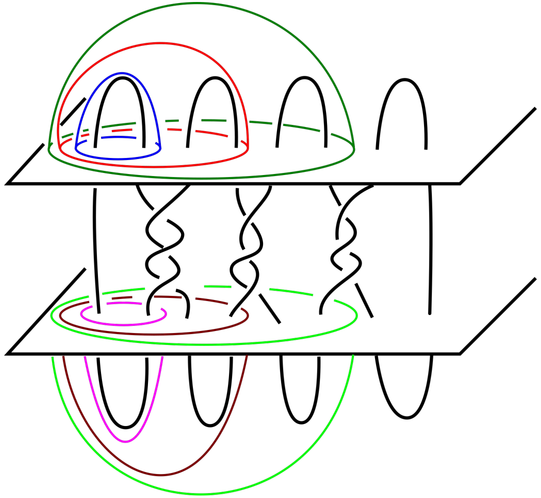


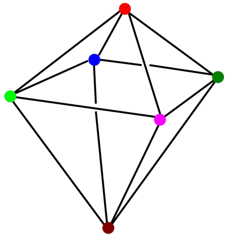
(P. Rodman) connected, but not simply connected disk complex



(P. Rodman) simply connected, but not 2-connected disk complex

Theorem (P., Rodman): There is an infinite family of bridge surfaces with simply connected, but not 2-connected disk complex.





Connected &
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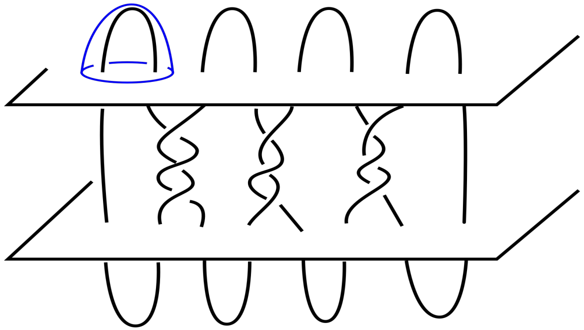

Can color each disk
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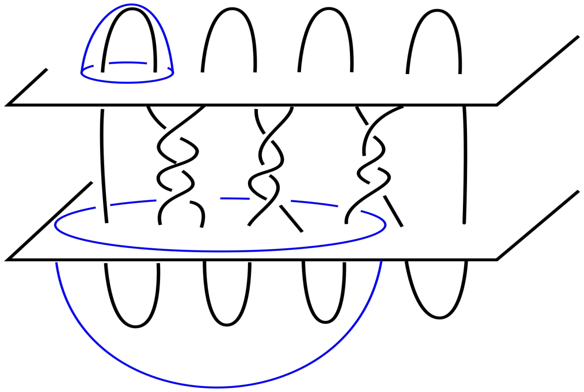
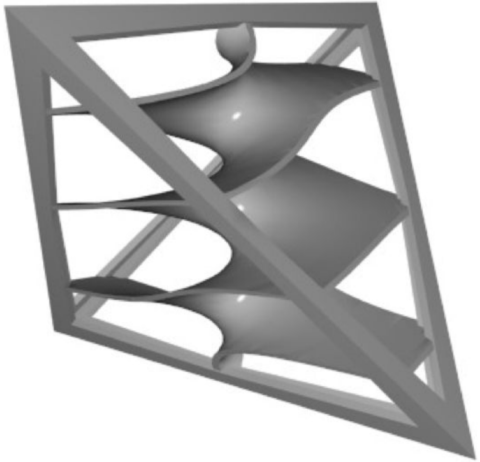


Thank you!



Surfaces with non-contractible disk complexes behave like minimal surfaces.

Bachman, David & Derby-Talbot, Ryan & Sedgwick, Eric. (2015). Locally Helical Surfaces have bounded twisting. Pacific Journal of Mathematics. 292. 10.2140/pjm.2018.292.257.

Obstructions to the Existence of Lagrangians in $\mathbb{R}^4$

Ipsita Datta

Stanford University

ipsi@stanford.edu

Set up

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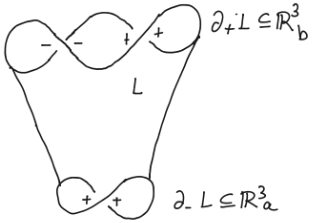
and standard complex structure: $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$, $i(\partial_{x_j}) = \partial_{y_j}$.

Lagrangian cobordism between links

Consider an exact Lagrangian surface $L \subset \{a \leq y_2 \leq b\} \subset \mathbb{R}^4$ which is



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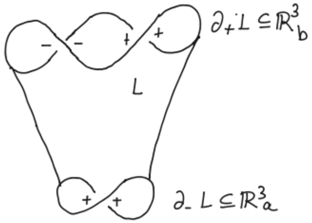

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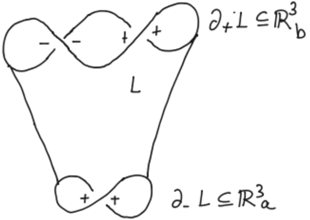

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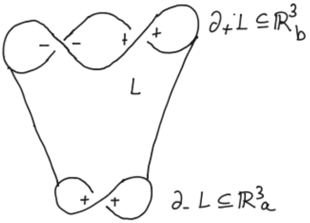
So, its boundaries are links $\partial_- L$ and $\partial_+ L$ in copies of $\mathbb{R}^3$.



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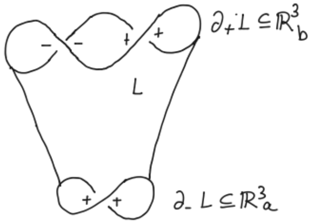

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L is an exact Lagrangian cobordism between embedded links. We write

$$\partial_- L \prec \partial_+ L.$$

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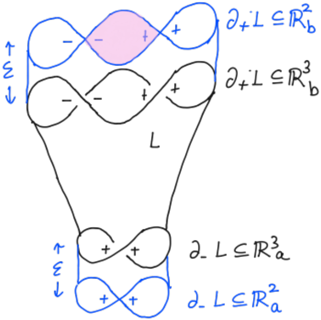

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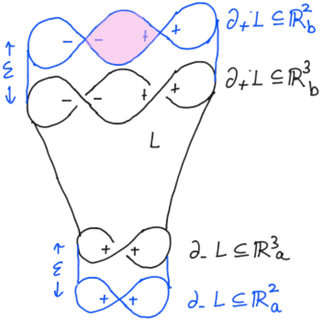

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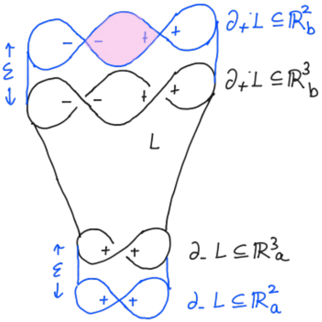

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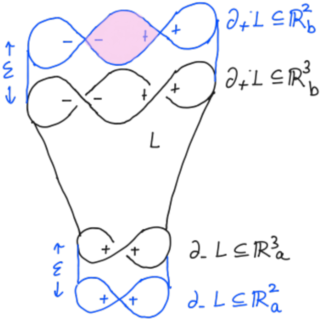

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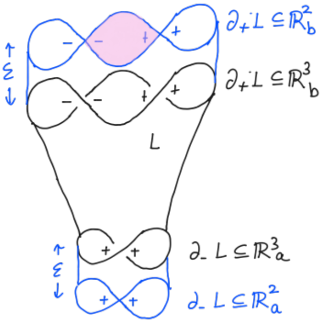

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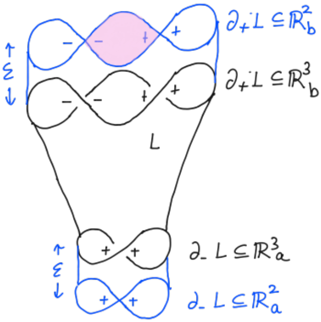

Caterpillar C^{+++}

Theorem (D.)

The caterpillar knot C^{+++} cannot be the boundary of an exact Lagrangian surface $L \subset \{y_2 \leq a\}$ with $C^{+++} \subset \{y_2 = a\}$.

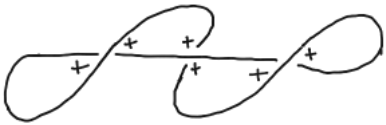

Figure: C^{+++}

This answers a question from *A Partial Ordering on Slices of Planar Lagrangians* by P. Eiseman, J. Lima, J. Sabloff, and L. Traynor.

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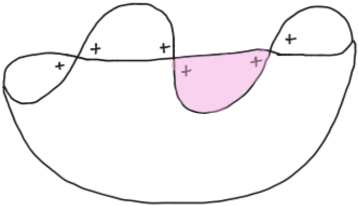

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There are no possible other boundary points from "sign conditions".

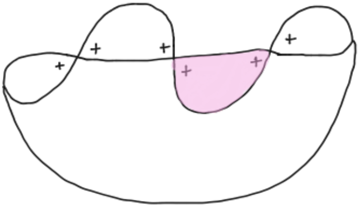

Figure 8-knots

Theorem (Sabloff, Traynor)

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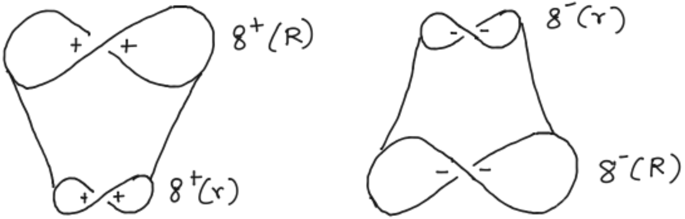
- If $8_-(R) \prec 8_-(r)$, then $R > r$.



Originally shown in *Obstructions to the Existence and Squeezing of Lagrangian Cobordisms* by J. Sabloff, and L. Traynor using generating functions. We reprove this using holomorphic curves.

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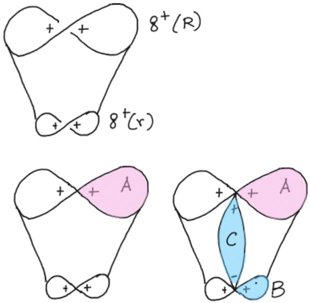

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- If $8_-(R) \prec 8_-(r)$, then $R > r$.

Disk A is a boundary
point of a 1-dimensional moduli space.

Possible other boundaries
look like the degenerate disk (C, B).

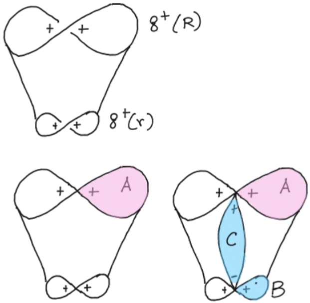

Figure 8-knots

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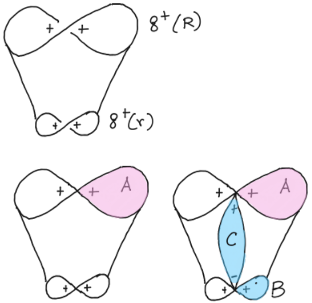

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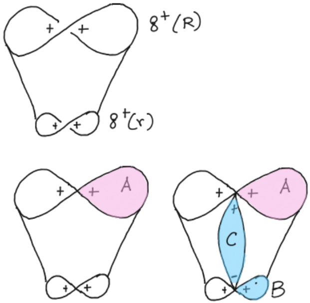

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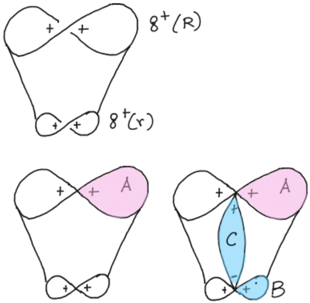
So, area $B <$ area A.



Homology Concordance and an Infinite Rank Subgroup

Hugo Zhou

Georgia Tech

Backgrounds

Two knots in S^3 are **smoothly concordant** if they cobound a smooth annulus in $S^3 \times I$.

Definition

Knot concordance group $\mathcal{C} := (K, \#)/$ smooth concordance.

J. Levine (69.) proved that there is a surjective homomorphism from knot concordance group $\mathcal{C}$ to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$. In particular, $\mathcal{C}$ contains a $\mathbb{Z}^\infty$ summand.

Backgrounds

Definition

Suppose knots $K_0 \subset Y_0$, $K_1 \subset Y_1$, Y_0 and Y_1 are homology cobordant.
K_0 and K_1 are **homology concordant** if they cobound a smooth annulus in some homology cobordism between Y_0 and Y_1.

Backgrounds

Definition

Let $\widehat{\mathcal{C}}_{\mathbb{Z}} := \big((Y, K), \#\big)/$ homology concordance, where Y is a homology 3-sphere that is homology cobordant to S^3.

Definition

Let $\mathcal{C}_{\mathbb{Z}} := \big((S^3, K), \#\big)/$ homology concordance.

The quotient group $\widehat{\mathcal{C}}_{\mathbb{Z}}/\mathcal{C}_{\mathbb{Z}}$ measures the "difference" between knots in S^3 and knots in homology spheres.

Question:

Does $\widehat{\mathcal{C}}_{\mathbb{Z}}/\mathcal{C}_{\mathbb{Z}}$ contain a $\mathbb{Z}^\infty$ summand?

- Adam Simon Levine (14.) showed that $\widehat{\mathcal{C}}_{\mathbb{Z}}/\mathcal{C}_{\mathbb{Z}}$ is not trivial;
- Hom, Levine, Lidman (18.) proved that $\widehat{\mathcal{C}}_{\mathbb{Z}}/\mathcal{C}_{\mathbb{Z}}$ is infinitely generated and contains a $\mathbb{Z}$ subgroup.

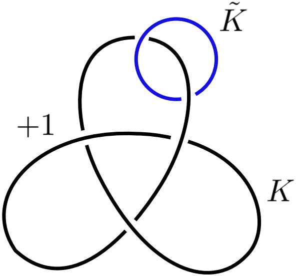
Main Result

Main Theorem (Z.)

$\widehat{\mathcal{C}}_{\mathbb{Z}}/\mathcal{C}_{\mathbb{Z}}$ contains a $\mathbb{Z}^\infty$ subgroup.

1. Infinitely many generating pairs constructed by applying the **filtered mapping cone formula (Hedden, Levine)** on the **L-space knots**.
2. Linearly independence proved using **connected knot complex**.

The filtered mapping cone formula computes $CFK^{\infty}(S^3_1(K), \tilde{K})$.



The knot K in this figure is the right handed trefoil .

Computational result

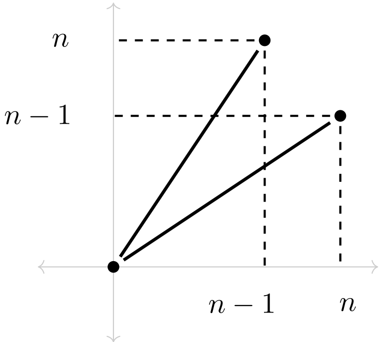


Figure: Connected knot complex of $CFK^\infty(S^3_1(T_{2,4n-1}), \widetilde{T}_{2,4n-1})$

Technical point: Connected sum with the unknot in $-S^3_1(T_{2,4n-1})$ to complete the construction.

Rigidity on the Morse boundary

Qing Liu

Brandeis University

Tech Topology Conference

Dec. 4-6. 2020

Motivation:

Q: When a group G determined by its boundary ∂G ?

In the case of hyperbolic groups, we know the answer.

<u>Case I</u>: G is a hyperbolic group.

- X is a hyperbolic space,

$$\partial X = \{ \alpha \mid \alpha \text{ geodesic ray} \} / \sim$$

Thm (Gromov) X, Y are hyperbolic spaces.

If $f: X \to Y$ is a quasi-isometry,

then f induces a homeomorphism

$$F: \partial X \xrightarrow{\cong} \partial Y.$$

$\rightsquigarrow$ If G is a hyperbolic groups,

then ∂G is well-defined.

Thm (Paulin '96).

X, Y are proper, cocompact hyperbolic spaces.

Suppose $F: \partial X \to \partial Y$ is a homeomorphism.

Then TFAE:

(1) F is induced by a quasi-isometry $f: X \to Y$.

(2) F is quasi-mobius.

(3) F is quasi-conformal.

Thm (Bonk and Schramm):

X, Y are hyperbolic spaces.

If $F: (\partial X, d_{x_0,\varepsilon}) \to (\partial Y, d_{y_0,\varepsilon'})$ is a power quasisymmetry,

then F extends to a quasi-isometry $f: X \to Y$.

The Gromov boundary ∂X of a hyperbolic space X is metrizable !!

Next: Can we do the same for a f.g. group G?

Case II: G is a f.g. group.

- X is proper geodesic metric space,

$$\partial_M X = \{ \alpha \mid \alpha \text{ is Morse geodesic ray} \} / \sim$$

↑ the Morse boundary.

- proper CAT(0) space: Charney and Sultan.
- proper geodesic metric space: Cordes.

- $\partial_M G$ is well-defined for any f.g. group G.

but it is not metrizable in general.

Thm (Charney, Cordes, Murray 19).

X, Y. proper cocompact geodesic metric spaces. $|\partial_M X| \geq 3$.

Let $h: \partial_M X \to \partial_M Y$ be a homeomorphism.

Then h is induced by a quasi-isometry $f: X \to Y$ if and only if h and h^{-1} are 2-stable and quasi-mobius

<u>Thm</u> (Liu)

X. Y. proper cocompact geodesic metric spaces. $|\partial_M X| \geq 3$.

let $h: \partial_M X \to \partial_M Y$ be a homeomorphism

Then TFAE:

(1) h is induced by a quasi-isometry $f: X \to Y$.

(2) h. h^{-1} are <u>biHölder</u>.

(3) h. h^{-1} are <u>quasisymmetry</u>.

(4) h. h^{-1} are <u>strongly quasi-conformal</u>.

Qing Liu. Rigidity on the Morse boundary.

Cor: G, H are f.g. groups.

$|\partial_M G| \geq 3$. Let $h: \partial_M G \to \partial_M H$ be a homeomorphism.

Then TFAE:

(1) h is induced by a QI $f: X \to Y$.

(2) h, h^{-1} are 2-stable and quasi-mobius.

(3) h, h^{-1} are biholder.

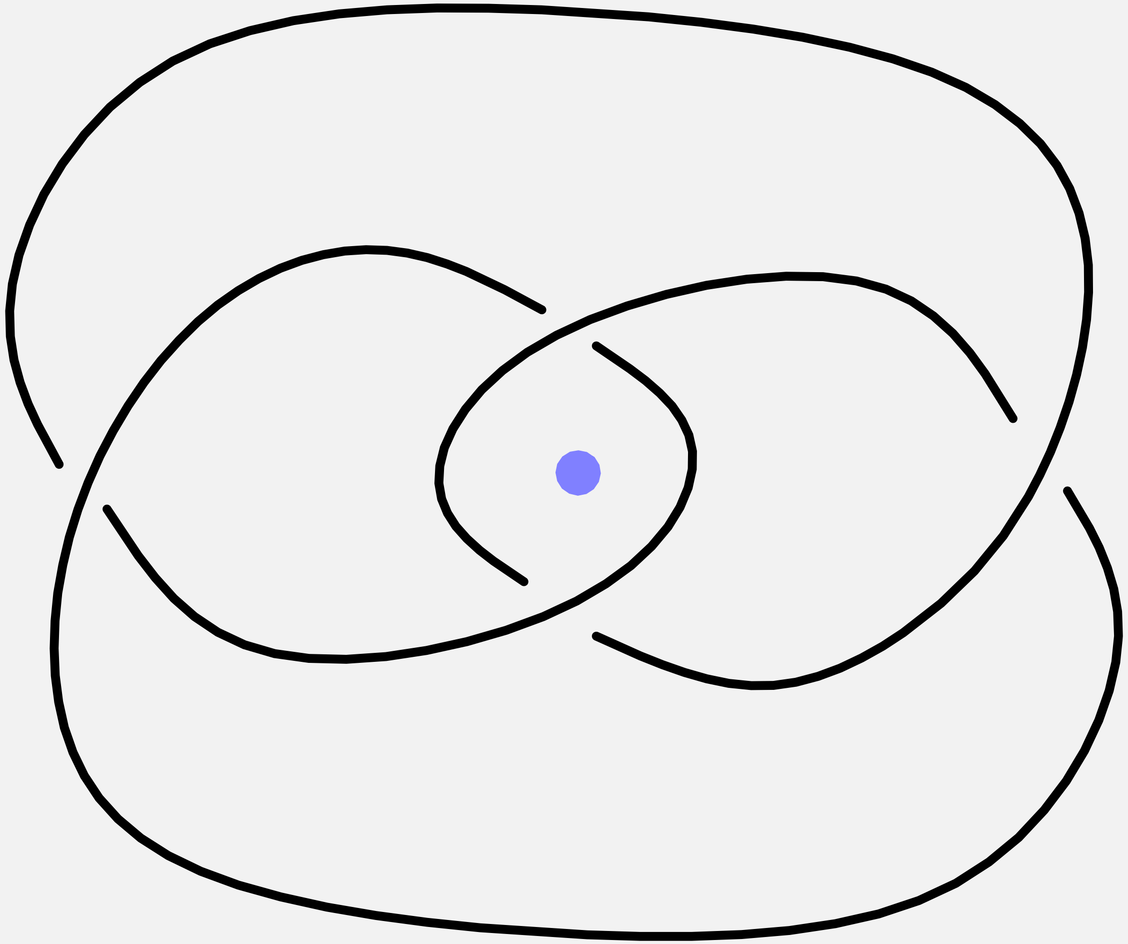
(4) h, h^{-1} are quasisymmetry.

(5) h, h^{-1} are strongly quasi-conformal.

Thank you!!

Visibility of symmetries, L-spaces, and branched cyclic covers

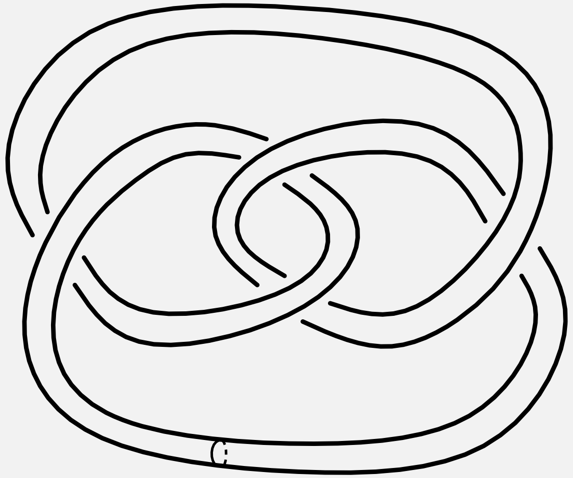
Hannah Turner



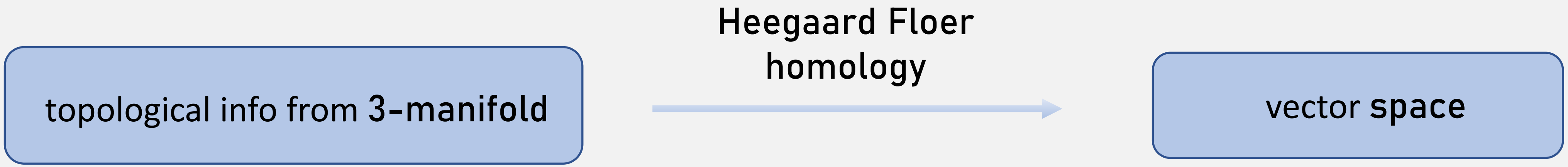
Branched cyclic covers

index n (cyclic) cover of the complement

branched (cyclic) cover of index n

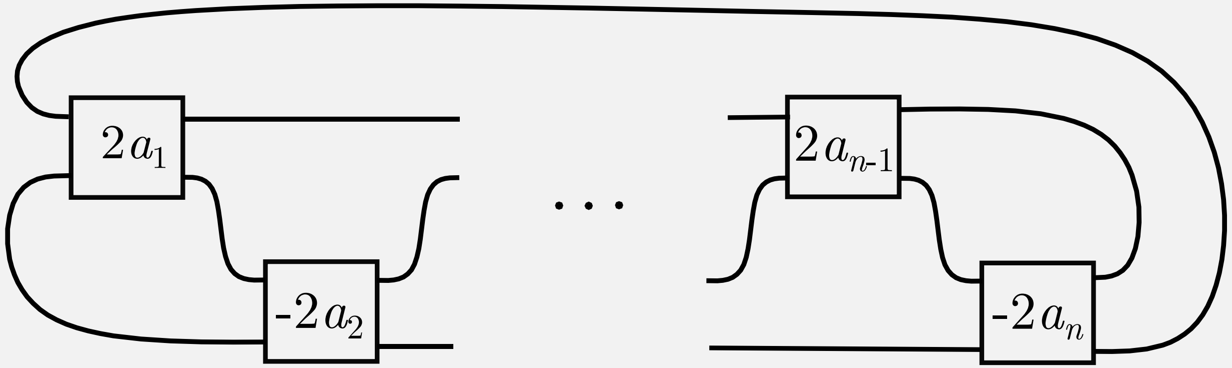


L-spaces

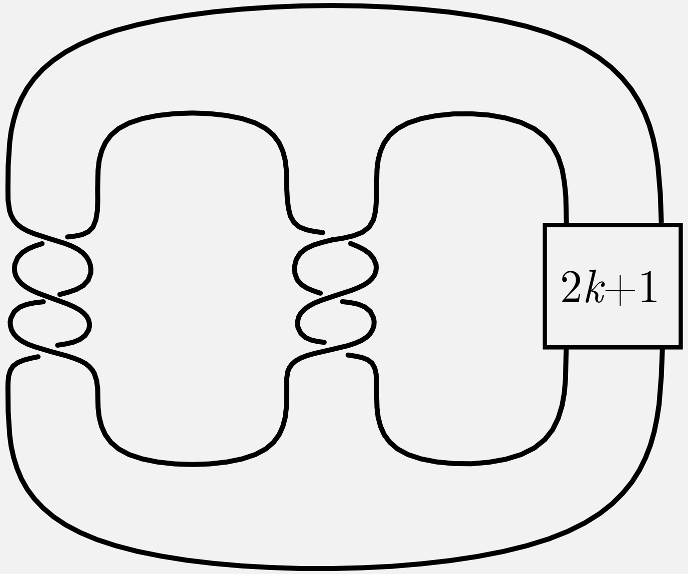


An L-space is a 3-manifold with “simple” Heegaard Floer homology

For which knots is every cyclic branched cover an L-space?

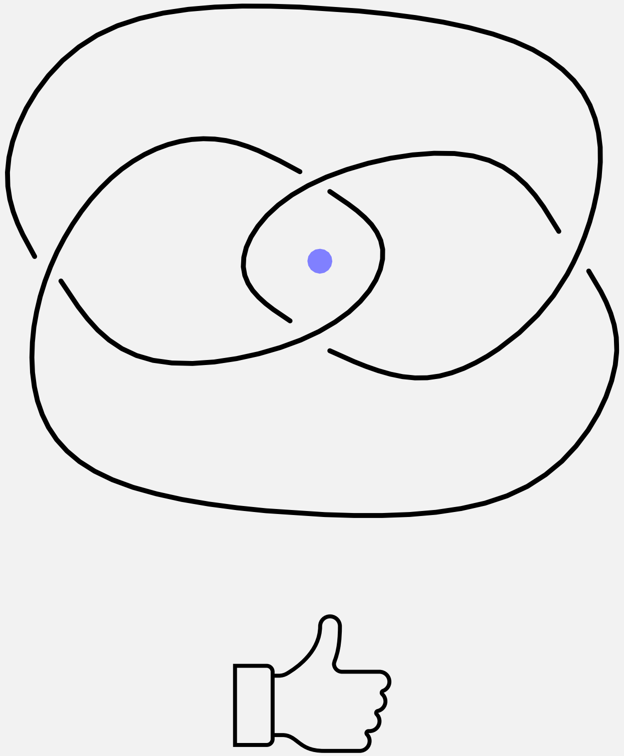


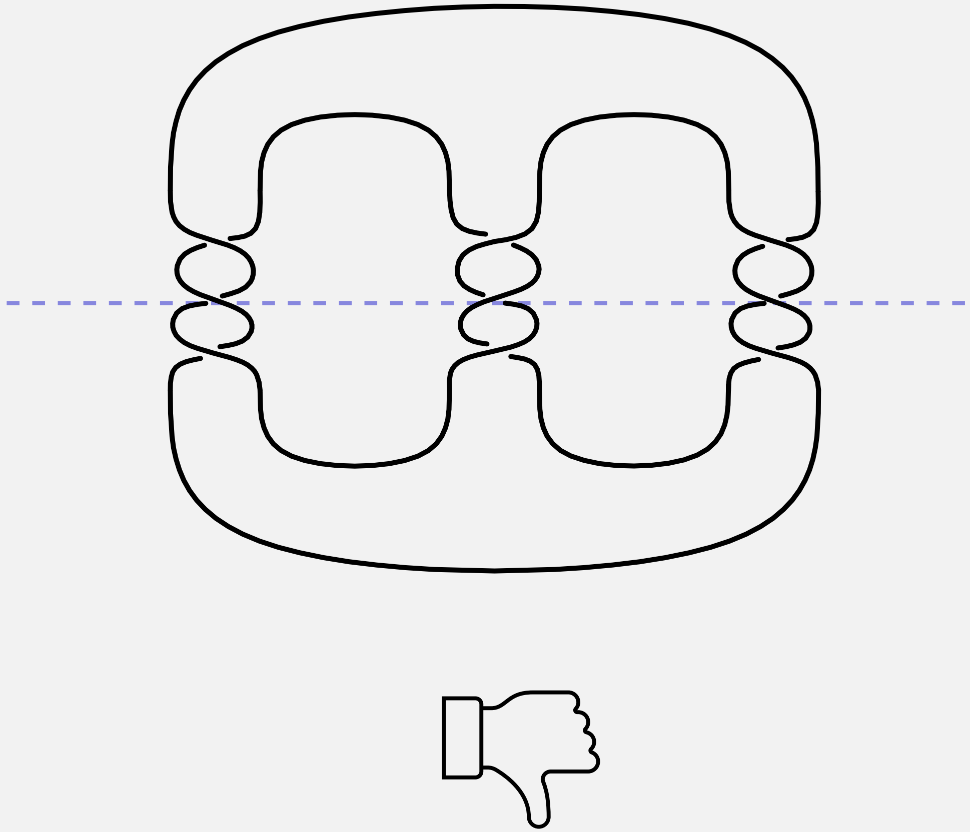
Peters, Teragaito



Issa-T.

Visibility of symmetries





Alternating knots and visibility

<u>Theorem</u> (Costa – Van Quach Hongler)

Let K be a prime alternating which has an order n symmetry. If n is at least 3, this symmetry is visible in some alternating diagram of K.

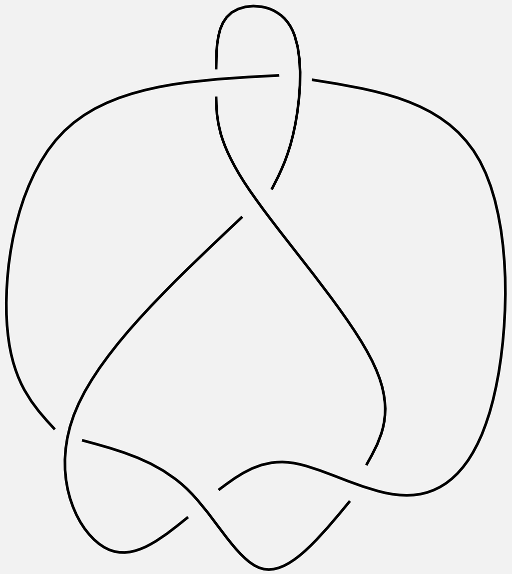
↓

<u>Theorem</u> (Paoluzzi)

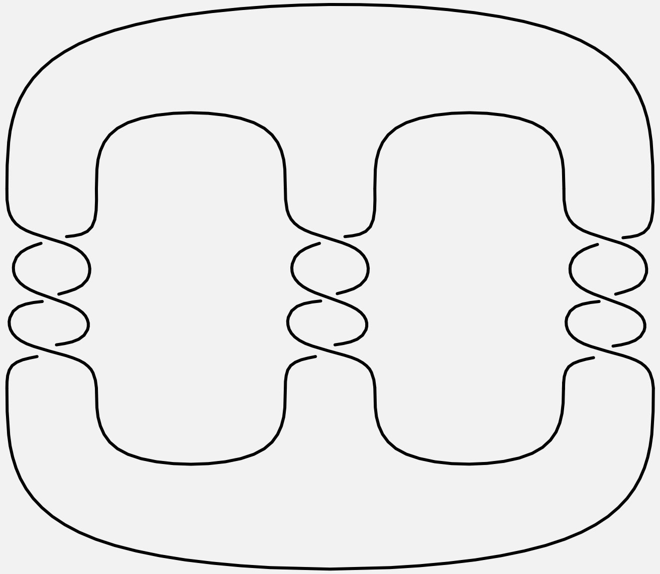
Let K be alternating. Then cyclic branched covers of K, of index n at least 3, each determine K.

What about index 2?

<u>Theorem</u> (T.) Let K be a prime alternating knot with an order 2 symmetry to the unknot. Then the symmetry is visible in an alternating diagram only if all of the cyclic branched covers of K are L-spaces.



alternating	✓
order 2 symmetry to the unknot	✓
visible in an alternating diagram	✗



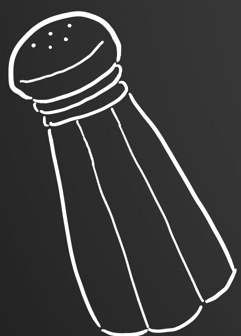
Υ-LIKE INVARIANTS FROM KHOVANOV HOMOLOGY

MELISSA ZHANG (UGA)

Joint ongoing work with

ROSS AKHMECHET (UVA)

↑ grad student on the job market this year!



LINK CONCORDANCE AND HOMOLOGY-TYPE INVARIANTS

Knot K $\longrightarrow$ 3-manifold Y^3

- 3D point of view
- quite ... geometric

Very exciting!
Much structure!

Knot K $\longrightarrow$ $Y^3 = \partial W^4$

- 4D point of view
- quite ...
- Knots / concordance is a group!
  (4D equivalence relation)

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$$K \hookrightarrow S^3 = \partial B^4$$

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filtered chain homotopy type $\rightsquigarrow$ total homology is invariant of ambient space

extract numer[illegible] invariants

- bounds on 3-ball genus, 4-ball genus
- understand the concordance group
- ...

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extract numerical concordance invariants

- bounds on 3-ball genus, 4-ball genus
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EXISTING INVARIANTS

- filtration grading of distinguished homology class
- grading of nontorsion tower
- extremal level where induced map is nonzero

CFK^-:
- τ [Ozsváth-Szabó, Rasmussen]
- h_i [Rasmussen]
- Υ [Ozsváth-Stipsicz-Szabó]
- Υ^2 [Kim-Livingston]
- Υ^c [Alfieri]

$\mathbb{F}[U,V]/(UV)$:
- φ_j [Dai-Hom-Stoffregen-Truong]

surgery:
- ν [Ozsváth-Szabó]
- ν^+, ν^- [Hom-Wu]
- ν_n [Truong]
- γ_4 [Golla-Marengon]

* for nonorientable slice genus bound
not to be confused
with genus bound from ε. [Hom'12]

Khovanov (sl_2) - Lee / Bar-Natan homology:
- s [Rasmussen]

sl_n homology:
- λ [Lewark-Lobb]

Sarkar-Seed-Szabó's perturbation of Szabó's geometric spectral sequence:
- s^u [Sarkar-Seed-Szabó]
- $s_{r,t}$ [Truong-Z]

annular concordance:
- $s_{r,t}$ [Truong-Z]
- d_t [Grigsby-Licata-Wehrli]

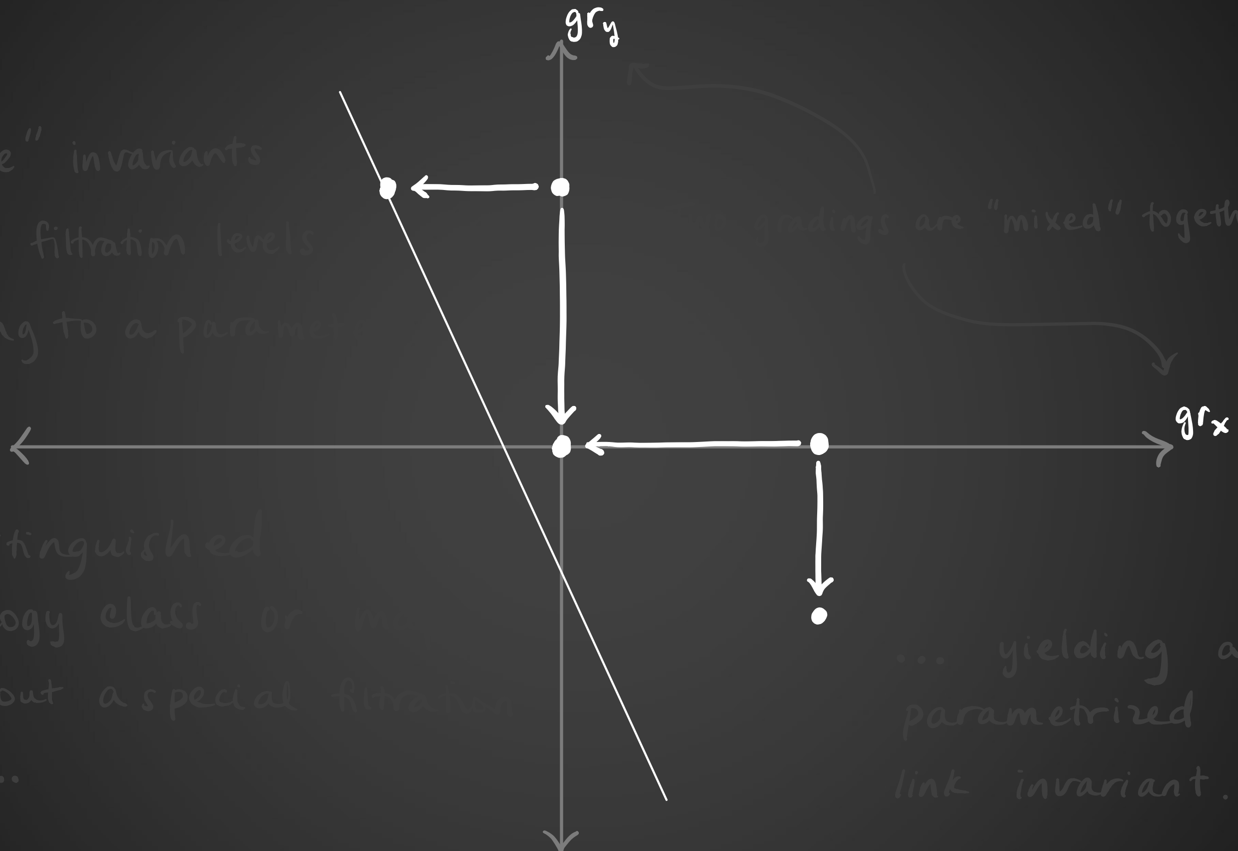
double branched cover:
- δ [Owens-Manolescu, Jabuka]
- $\underline{\delta}, \bar{\delta}$ [Alfieri-Kang-Stipsicz]

involutive:
- $\underline{\delta}, \bar{\delta}$ [Alfieri-Kang-Stipsicz]
- $\underline{V}_0, \bar{V}_0$ [Hendricks-Manolescu]

Sources: Hom's survey "A survey on Heegaard Floer Homology and Concordance"
Celoria's slides "Some concordance invariants from Knot Floer homology"

⚠ This is certainly incomplete. If you have any additions or corrections, let me know!

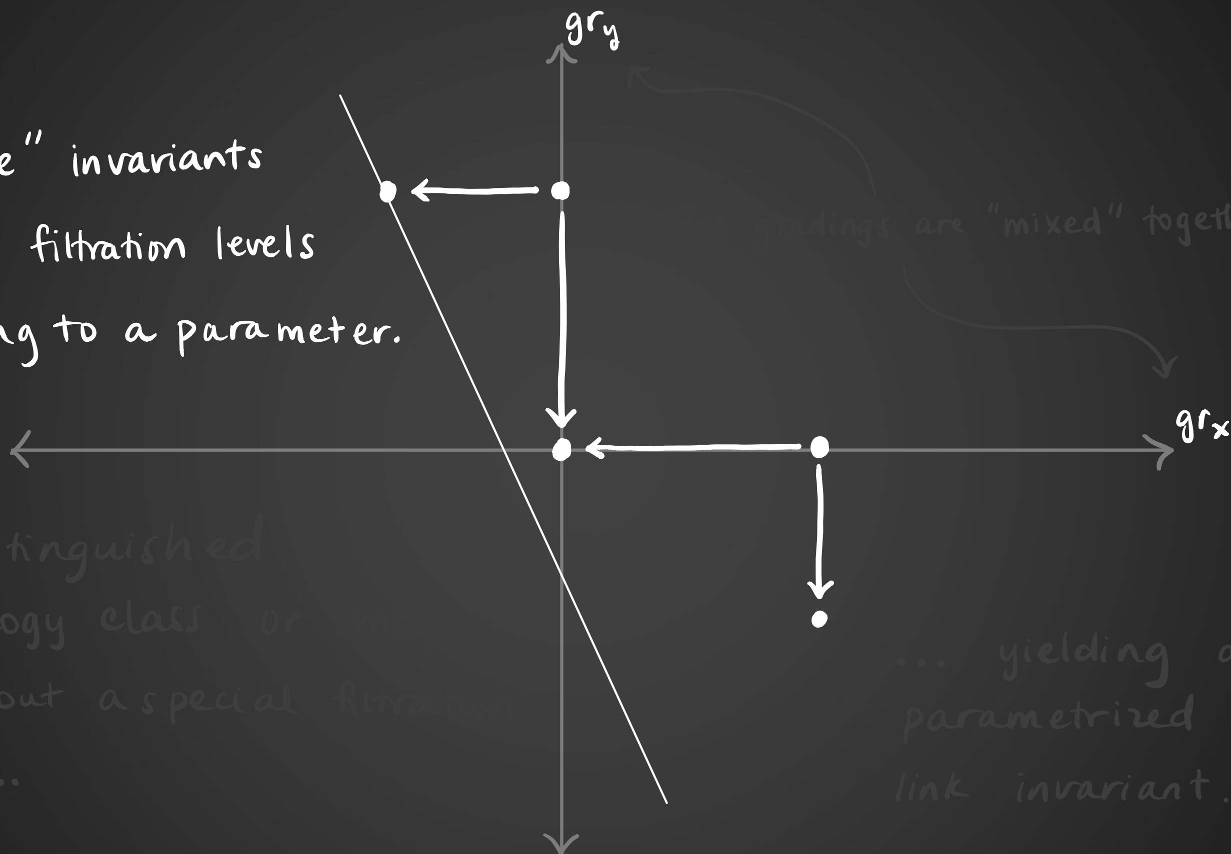
EXISTING INVARIANTS



See Livingston's "Notes on the knot concordance invariant Upsilon"

EXISTING INVARIANTS

"Υ-like" invariants tilt the filtration levels according to a parameter.



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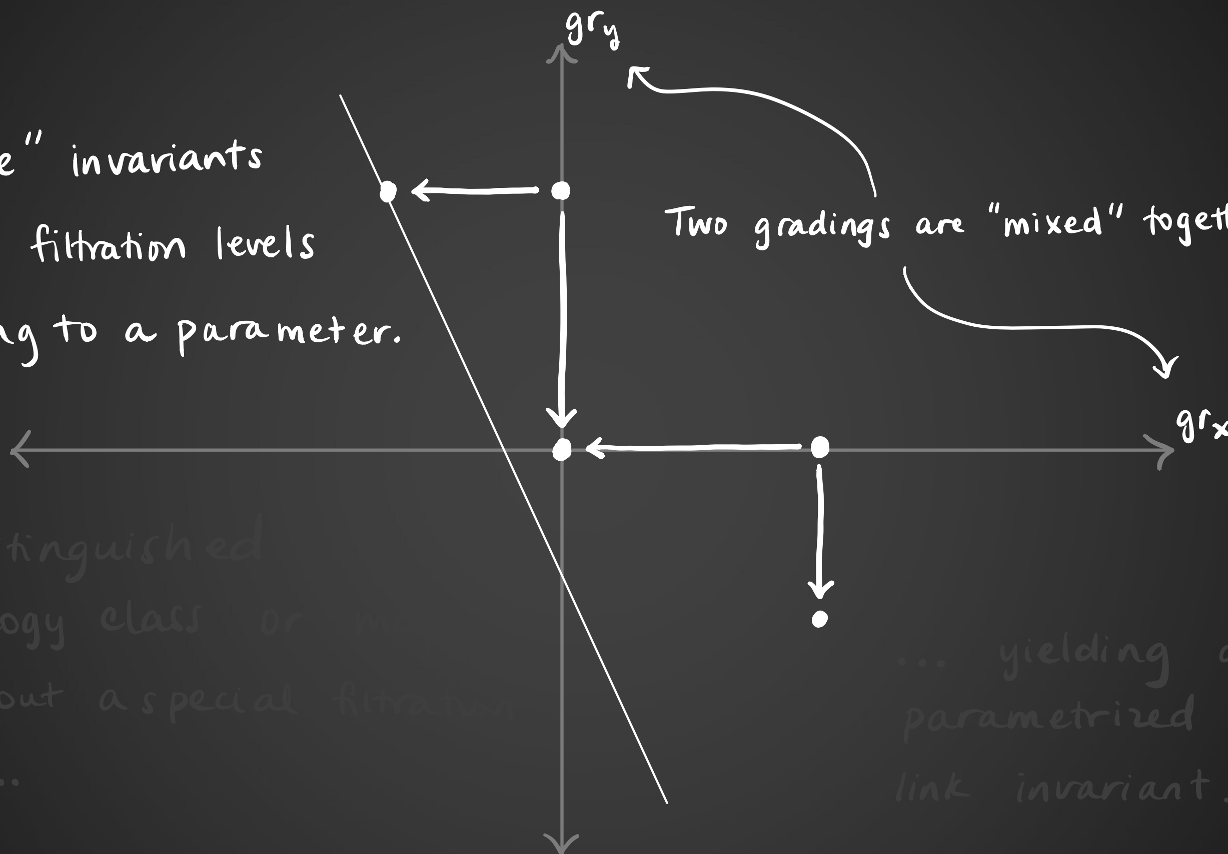

A distinguished homology class or m... picks out a special filtration level...

... yielding a parametrized link invariant.

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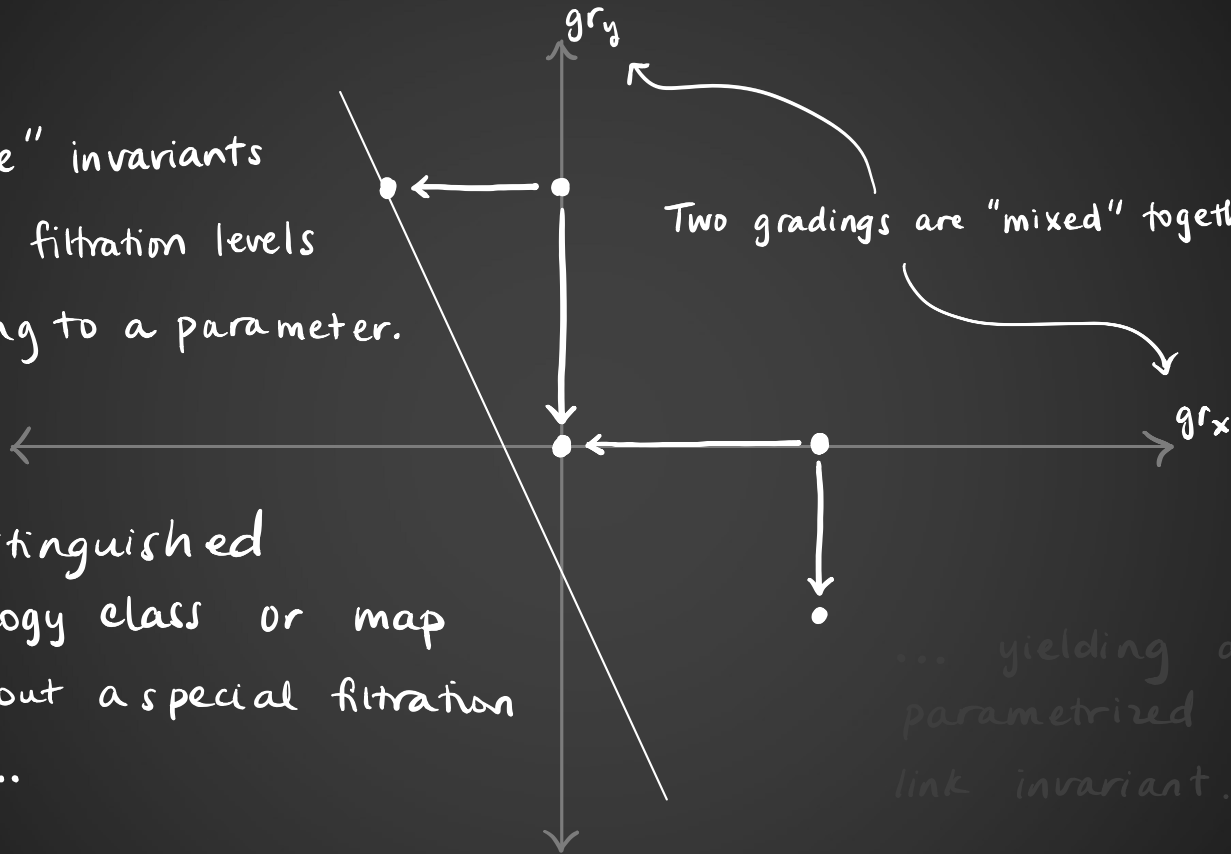

Two gradings are "mixed" together.

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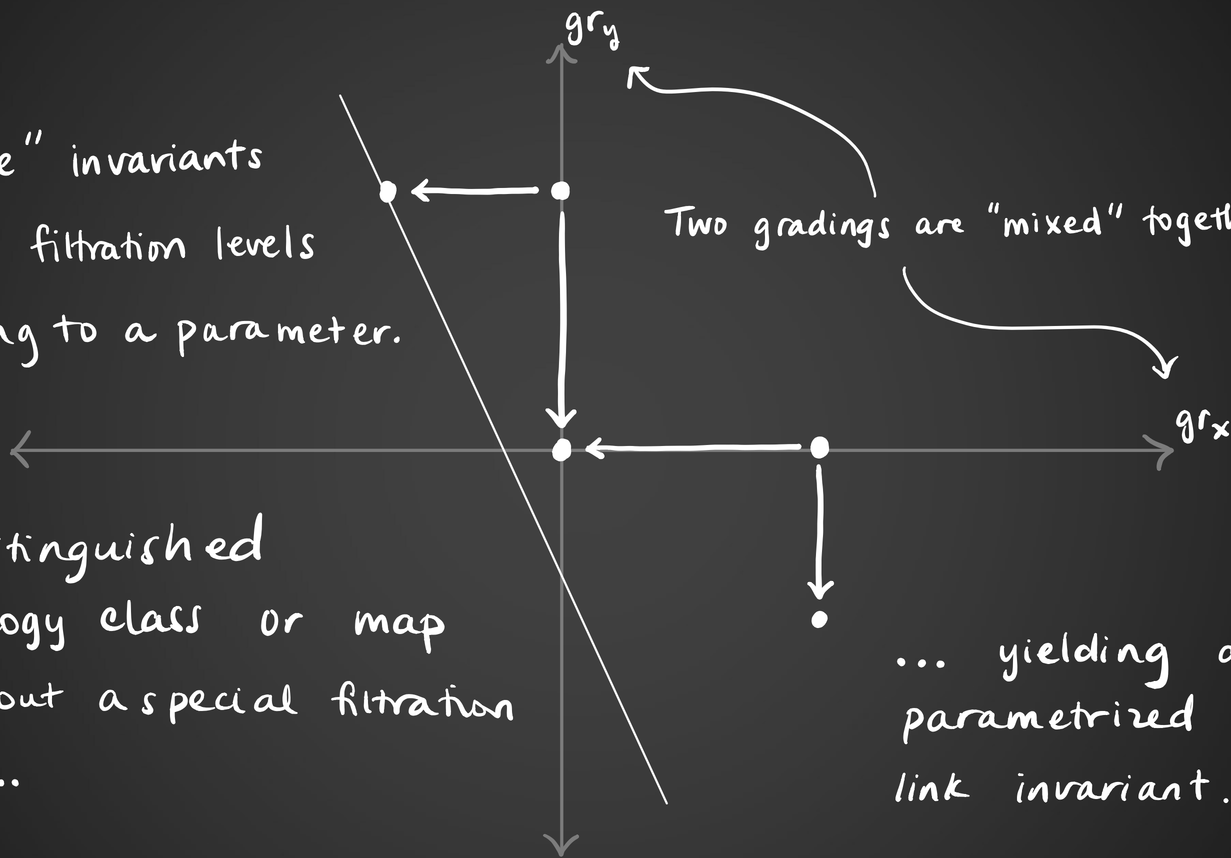

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U(1) × U(1) - EQUIVARIANT KHOVANOV HOMOLOGY

Kh is a functor

$$(\text{Links, Cobordisms}) \xrightarrow{Kh} \mathbb{Z}\oplus\mathbb{Z}\text{-graded } R\text{-module}$$

where R = coefficient ring [Khovanov '00]

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$U(2)$-equivariant Kh uses the Frobenius algebra

$$\mathcal{A} = R[X]/(X^2 - hX + t)$$

a.k.a. "universal Khovanov homology"
[Khovanov '04; studied by many others]

eg. $R = \mathbb{Q}, \mathbb{C}$; … homology ⇝ s-invariant
[Rasmussen ~'04]

$U(1) \times U(1)$ equivariant … Robert '18]

$$\mathcal{R}_\alpha = \mathbb{Z}[\alpha_1, \alpha_2] \qquad \mathcal{A}_\alpha = R[X]/(X-\alpha_1)(X-\alpha_2)$$

Not an integral domain!

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[Lee ~'02] [Rasmussen ~'04]

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A GENERALIZATION OF RASMUSSEN'S s-INVARIANT

Fix $t \in [0, 4]$.

t-modified $U(1) \times U(1)$ Khovanov Homology

$$\left(CKh^t_\alpha(D), \partial^t_\alpha(D) \right)$$

[Akhmechet - Z, work in progress]

A GENERALIZATION OF RASMUSSEN'S s-INVARIANT

Fix $t \in [0, 4]$.

Actually the ring of long power series (Removes dependence on N)

t-modified $U(1) \times U(1)$ Khovanov Homology

$$\left.\begin{array}{l} \varphi_t : R_\alpha \longrightarrow \mathbb{Z}[v^{1/N}] \\ \quad \alpha_1 \longmapsto v^t, \ \alpha_2 \longmapsto -v^{4-t} \\ \mathcal{A} = \mathcal{A}_\alpha \otimes \mathbb{Z}[v^{1/N}] \end{array}\right\} \rightsquigarrow \left(CKh^t_\alpha(D), \ \partial^t_\alpha(D) \right)$$

$gr_q(v) = -1$

[Akhmechet - Z, work in progress]

The differential is [illegible] with respect to a grading gr_t that "combines" [illegible] α_1- and α_2- [illegible].

The total homology [illegible] 2 non-v-[illegible] towers.

- concordance [illegible] : $s^t_\alpha(K)$
- computations to determine effectiveness, obstructions
- relationship with other invariants

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The total homology has 2 non-v-torsion towers.

- concordance invariant for $L \subset S^3$: $s^t_\alpha(K)$
- computations to determine effectiveness, obstructions
- relationship with other invariants

Thank You!

Exotic 4-manifolds with boundary

Hyunki Min
with John Etnyre and Anubhav Mukherjee

Tech topology conference
December 2020

Exotic 4-manifolds with boundary

- A 4-manifold X admits exotic smooth structures if X admits more than one smooth structures.

Exotic 4-manifolds with boundary

- A 4-manifold X admits exotic smooth structures if X admits more than one smooth structures.

- A 3-manifold Y admits exotic fillings if Y bounds a compact 4-manifold X such that X admits more than one smooth structures.

Which 3-manifolds admit exotic fillings?

Previous results

- (Gompf) $\Sigma(2,3,6n-1)$

Which 3-manifolds admit exotic fillings?

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- (Hayden-Mark-Piccirillo) Boundary of exotic Mazur manifolds

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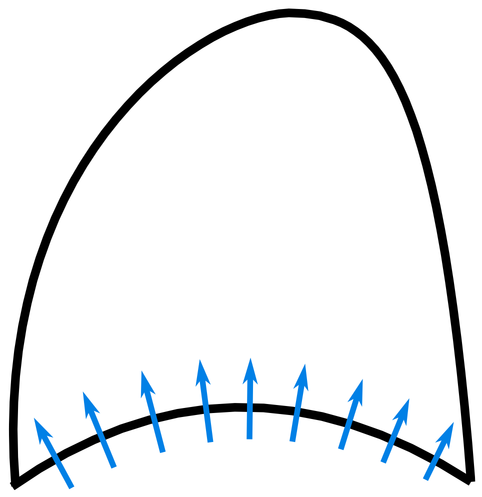
4) a rational homology 3-sphere embedding into a closed definite 4-manifold.

Exotic fillings

- Build a simply connected 4-manifold with boundary Y.



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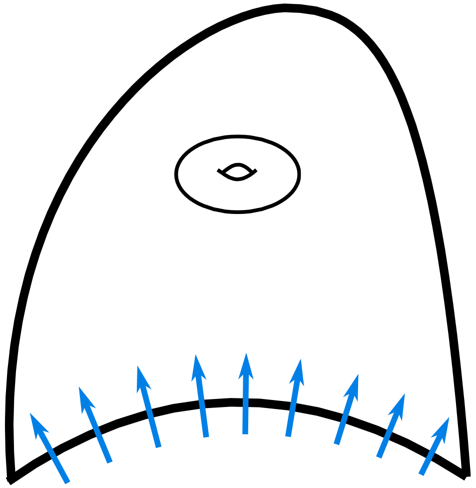
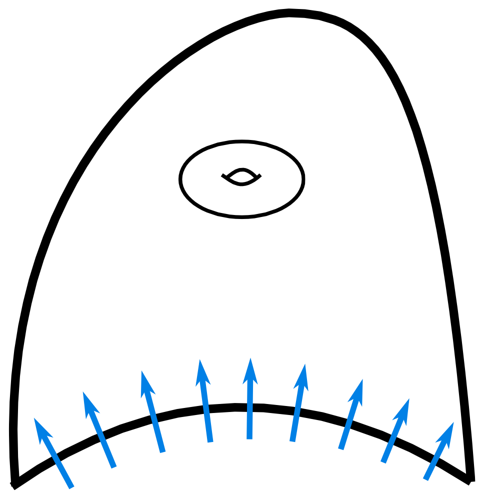
- Embed a torus with non-vanishing homology class and the 0 self intersection number.



Exotic fillings

- Build a simply connected 4-manifold with boundary Y.
- Embed a torus with non-vanishing homology class and the 0 self intersection number.
- Perform knot surgery on the torus and produce homeomorphic, but not diffeomorphic manifolds.



Thank you!