Nielsen realization problems

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Mapping class groups

Definition of $\mathrm{Mod}(M)$

M smooth manifold

Definition (mapping class group):

$$\mathrm{Mod}(M) = \{\, f : M \to M \text{ diffeomorphism}\}/\text{isotopy}$$

$$= \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$$

Diffeomorphisms isotopic to the identity

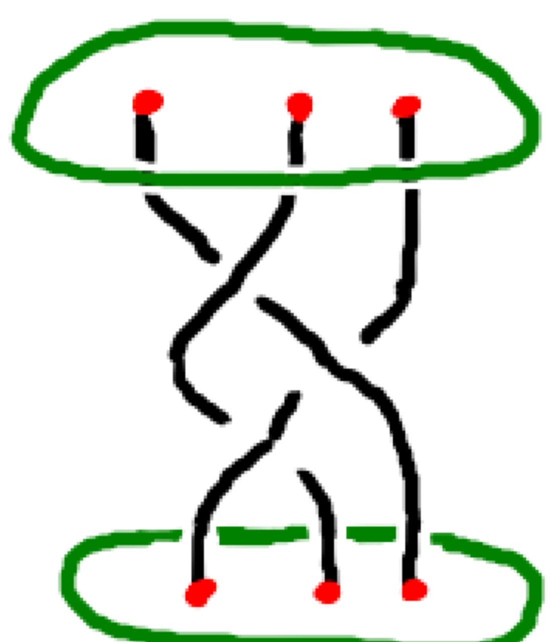
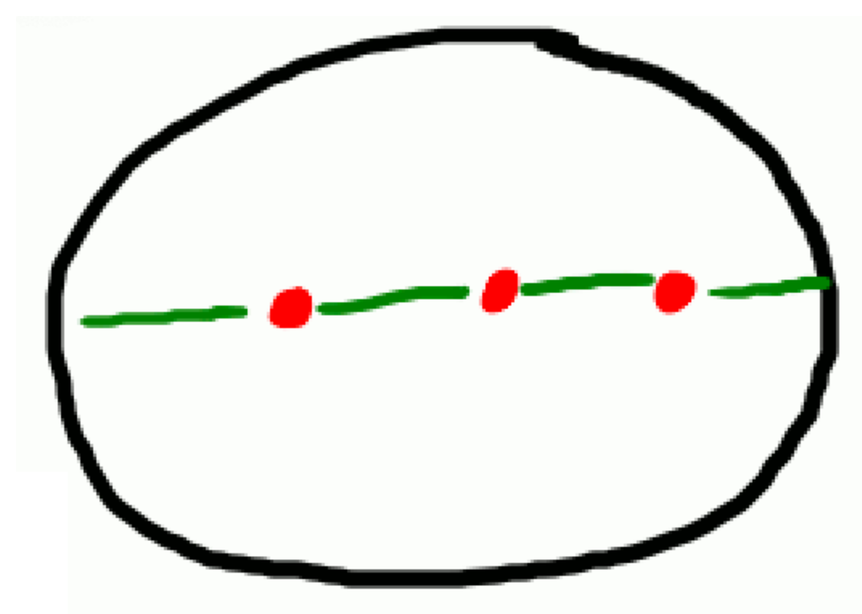
$$= \pi_0(\mathrm{Diff}(M))$$

Quick Examples

$$\mathrm{Mod}(M) := \pi_0(\mathrm{Diff}(M))$$

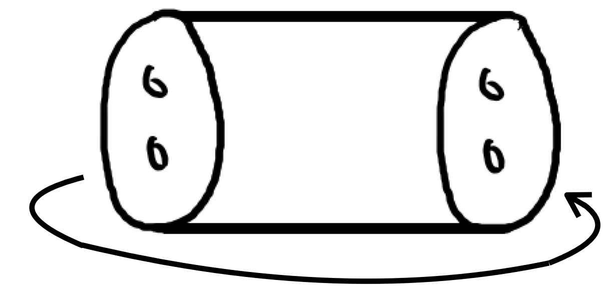
- $\mathrm{Mod}(T^2) \cong \mathrm{GL}_2(\mathbb{Z})$
- $\mathrm{Mod}(S^n) \cong \mathbb{Z}/2\mathbb{Z} \times \Theta_{n+1}$ ← $\{\text{Smooth structures on } S^{n+1}\}/\sim$
- $\mathrm{Diff}_\partial(D^2, n)$ diffeos preserving $\{\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n+1}\}$

$\mathrm{Mod}(D^2, n) := \pi_0(\mathrm{Diff}_\partial(D^2, n)) \cong \mathrm{Br}_n$ braid group

Motivations for studying $\mathrm{Mod}(S_g)$

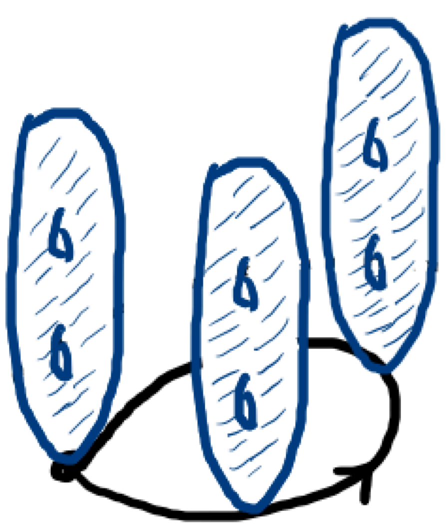
S_g closed oriented surface, genus g



$\varphi \in \mathrm{Diff}(S_g)$

1. Gluing data for 3- and 4-manifolds.
   - Mapping torus , Heegaard splitting, trisection
2. Moduli spaces/Teichmuller theory
   - $\mathrm{Mod}(S_g) \cong \pi_1(\mathcal{M}_g)$ moduli space of Riemann surfaces of genus g
3. Surface bundles

$$\left\{ \begin{array}{c} \text{fiber bundles} \\ S_g \to E \to B \end{array} \right\}_{/\text{iso}} \overset{\text{1-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{homomorphisms} \\ \pi_1(B) \to \mathrm{Mod}(S_g) \end{array} \right\}_{/\text{conj}}$$



$$(E \to B) \longmapsto \text{monodromy representation}$$

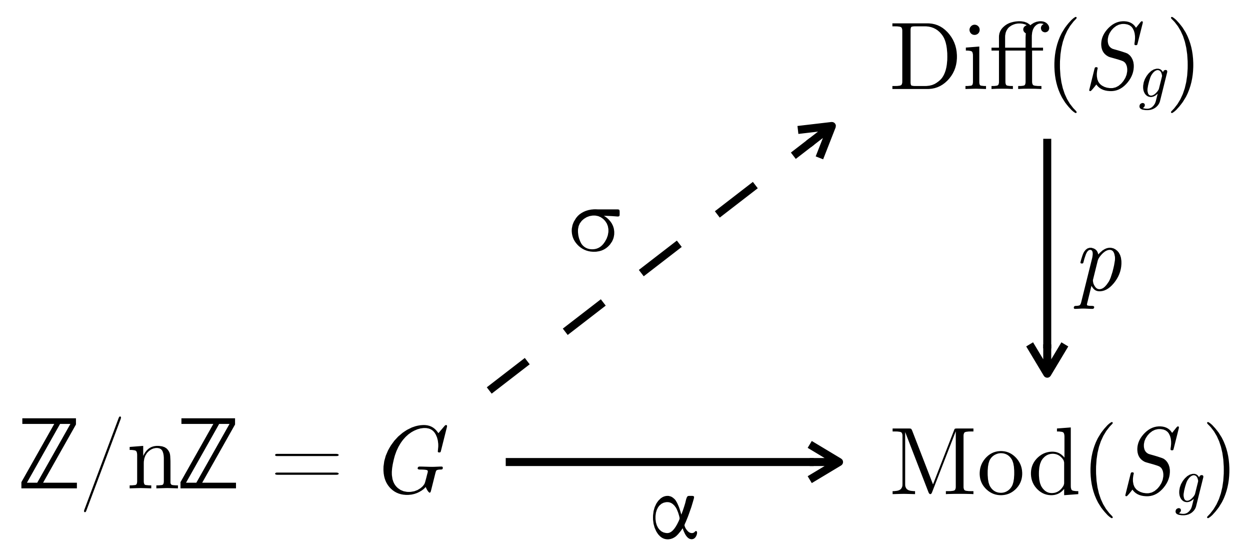
$$??? \longleftarrow \pi_1(B) \to \mathrm{Mod}(S_g)$$

Nielsen realization

Nielsen realization problem

Problem. Given α, does there exist a lift σ?

$$p \circ \sigma = \alpha$$



Originally asked by Nielsen (1906) for $G < \text{Mod}(S_g)$ finite.

E.g. if $F \in \text{Mod}(S_g)$ and $F^n = 1$, does there exist $\varphi \in \text{Diff}(S_g)$ such that $[\varphi] = F$ and $\varphi^n = \text{id}$?

(Kerckhoff, 1983): For every finite $G < \text{Mod}(S_g)$, a lift σ exists.

Realizing $G = \mathrm{Mod}(S_g)$

$$
\begin{array}{ccc}
 & & \mathrm{Diff}(S_g) \\
 & \overset{\sigma}{\nearrow} & \Big\downarrow p \\
\mathrm{Mod}(S_g) & \xrightarrow[\mathrm{id}]{} & \mathrm{Mod}(S_g)
\end{array}
$$

Question (Thurston). Does $p : \mathrm{Diff}(S_g) \to \mathrm{Mod}(S_g)$ split?

Example. Yes for $g{=}1$. $\quad \mathrm{Diff}(T^2) \overset{p}{\to} \mathrm{Mod}(T^2) \cong \mathrm{GL}_2(\mathbb{Z})$

Given $A{\in}\mathrm{GL}_2(\mathbb{Z})$, define $\sigma(A)$:

$$
\begin{array}{ccc}
\mathbb{R}^2 & \overset{A}{\to} & \mathbb{R}^2 \\
\Big\downarrow & & \Big\downarrow \\
T^2 = \mathbb{R}^2/\mathbb{Z}^2 & \xrightarrow[\sigma(A)]{} & \mathbb{R}^2/\mathbb{Z}^2
\end{array}
$$

Realizing $G = \mathrm{Mod}(S_g)$

Theorem (Morita, 1987). For $g \geq 10$,

$$p : \mathrm{Diff}^2(S_g) \to \mathrm{Mod}(S_g) \text{ does not split.}$$

Improvements:

- Markovic: homeomorphisms, $g \geq 5$ (2007)
- Franks-Handel: C^1 diffeos, $g \geq 5$ (2009)
- Bestvina-Church-Souto: C^1 diffeos, $g \geq 6$ (2013)
- Salter-Tshishiku: C^1 diffeos, $g \geq 2$ (2016)
- Chen, Salter-Chen: homeomorphisms, $g \geq 2$ (2020)

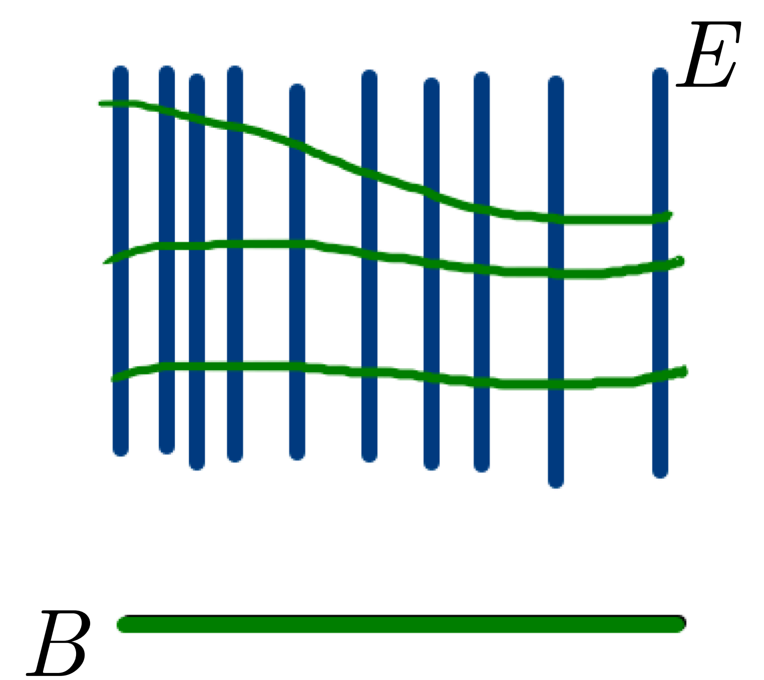
Nielsen realization and flat connections

Fix $S_g \to E \to B$
with monodromy
$\alpha : \pi_1(B) \to \mathrm{Mod}(S_g)$.

$$\begin{array}{ccc} & & \mathrm{Diff}(S_g) \\ & \overset{\sigma}{\nearrow} & \downarrow p \\ \pi_1(B) & \xrightarrow[\alpha]{} & \mathrm{Mod}(S_g) \end{array}$$

$$\left\{ \begin{array}{c} \text{fiber bundles} \\ S_g \to E \to B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{homomorphisms} \\ \pi_1(B) \to \mathrm{Mod}(S_g) \end{array} \right\}$$

$E \to B$ is *flat*, i.e. E has horizontal foliation $\iff$ $\alpha : \pi_1(B) \to \mathrm{Mod}(S_g)$ has a lift σ



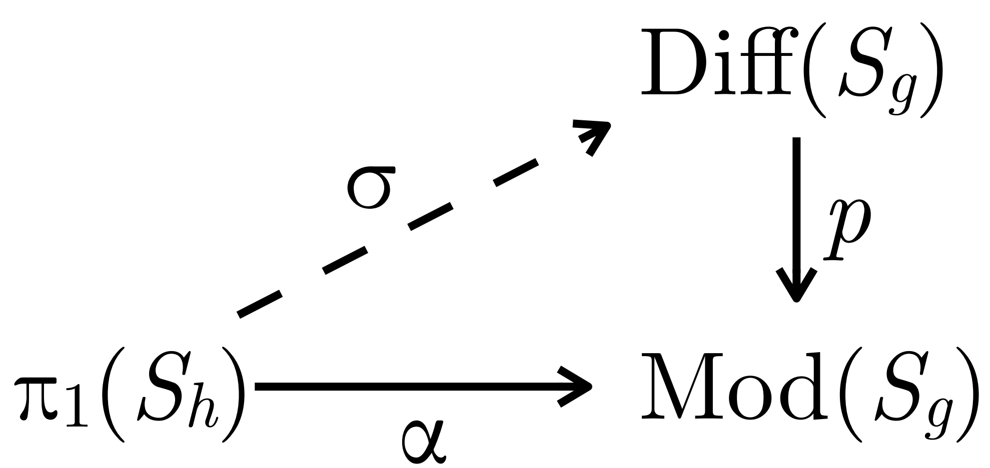
Nielsen realization and flat connections

Theorem (Morita, 1987). For $g \geq 10$,

$$p : \mathrm{Diff}^2(S_g) \to \mathrm{Mod}(S_g) \text{ does not split.}$$

Corollary (of Morita's Proof). For $g \geq 10$, there exists a bundle $S_g \to E \to B^6$ that's not flat.

Question. Does there exist $S_g \to E \to S_h$ that's not flat?



Nielsen realization and K3 manifolds

Theorem (Giansiracusa-Kupers-T). For K a K3 manifold,

$$p : \mathrm{Diff}(K) \to \mathrm{Mod}(K) \text{ does not split.}$$

$$\text{K3 manifold} \cong \{x^4+y^4+z^4+w^4=0\} \subset \mathbb{C}\mathrm{P}^3$$

Corollary. There exists a bundle $K \to E \to B^8$ that's not flat.

Proof Sketch

Theorem (Giansiracusa-Kupers-T).

$$p : \mathrm{Diff}(K) \to \mathrm{Mod}(K) \text{ does not split.}$$

Ingredients of the Proof (following Morita).

- Show $\mathrm{H}^*(\mathrm{Mod}(K);\mathbb{Q}) \xrightarrow{p^*} \mathrm{H}^*(\mathrm{Diff}(K);\mathbb{Q})$ not injective.

  characteristic classes — characteristic classes of flat bundles

- Bott vanishing: a certain characteristic class $\alpha \in \mathrm{H}^8(\mathrm{Mod}(K);\mathbb{Q})$ vanishes for flat bundles, i.e. $\alpha \in \ker(p^*)$.
- α obtained from $\beta \in \mathrm{H}^8(\mathrm{SO}(3,19;\mathbb{Z});\mathbb{Q})$

$$\mathrm{Mod}(K) \to \mathrm{Aut}(\mathrm{H}_2(K)) \simeq \mathrm{SO}(3,19;\mathbb{Z}).$$

$\beta \neq 0$: stable cohomology of arithmetic groups (Borel, Franke)

$\alpha \neq 0$: $\mathrm{H}^*(\mathrm{SO}(3,19;\mathbb{Z});\mathbb{Q}) \to \mathrm{H}^*(\mathrm{Mod}(K);\mathbb{Q})$ injective

moduli space of Einstein metrics (Giansiracusa) $\square$

Nielsen realization and exotic smooth structures

Symmetries of exotic smooth structures

$M = \mathbb{H}^n/\pi$ closed hyperbolic manifold

N exotic smooth structure: N,M homeo, not diffeo

e.g. $N = M \# \Sigma$ for $\Sigma \in \Theta_n$.

Question. How much symmetry does N have?

symmetry constant $\mathrm{s}(N) := \max \{ \, |G| : G < \mathrm{Diff}(N) \text{ finite}\}$

Example. For $\Sigma \in \Theta_n$, define

symmetry constant $\mathrm{s}(\Sigma) = \max \{\dim(G) : G < \mathrm{Diff}(\Sigma) \text{ Lie}\}$

(Hsiang-Hsiang). $n \gg 0$ and $\Sigma \neq S^n \Longrightarrow \mathrm{s}(\Sigma) < \frac{1}{4} \dim \mathrm{O}(n+1)$.

Connection to Nielsen realization

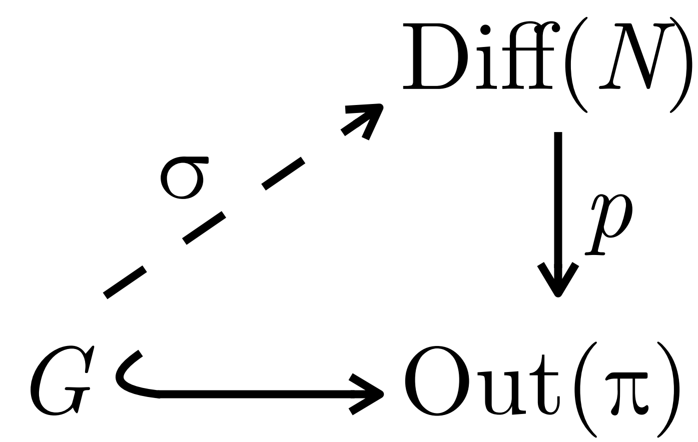
$M = \mathbb{H}^n/\pi$; N exotic smooth structure

Problem. Compute $s(N) := \max\{\,|G| : G < \mathrm{Diff}(N) \text{ finite}\}$

(Borel) $G < \mathrm{Diff}(N)$ finite $\Longrightarrow$ $G \hookrightarrow \mathrm{Diff}(N) \to \mathrm{Out}(\pi)$ injective.

Consequently, $1 \leq s(N) \leq |\mathrm{Out}(\pi)|$

and $s(N) = \max\{|G|: G < \mathrm{Out}(\pi)$ lifts



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- For $n = 2$, $\mathrm{Out}(\pi) \cong \mathrm{Mod}(M)$. (Dehn-Nielsen-Baer)
- For $n \geq 3$, $\mathrm{Out}(\pi) \cong \mathrm{Isom}(M)$. (Mostow rigidity)

Nielsen realization/exotic smooth structures

$M = \mathbb{H}^n/\pi$; N exotic smooth structure

$$1 \le s(N) := \max \{ |G| : G < \mathrm{Diff}(N) \text{ finite} \} \le |\mathrm{Out}(\pi)|$$

Theorem (Farrell-Jones). $\exists$ M, N so that $s(N) < |\mathrm{Out}(\pi)|$.

- They consider $N = M \# \Sigma$ for $\Sigma \in \Theta_n$.
- They show $\mathrm{Diff}(N) \to \mathrm{Out}(\pi)$ is not surjective.
- Consequently, $\mathrm{Diff}(N) \to \mathrm{Out}(\pi)$ does not split.

Theorem (Bustamante-T). For each $d \ge 2$, $\exists$ M, N so that $s(N) \le \frac{1}{d} |\mathrm{Out}(\pi)|$.

- The examples $N = M \# \Sigma$ do not work for $d \ge 3$.
- Show $\mathrm{Im}[\mathrm{Diff}(N) \to \mathrm{Out}(\pi)] < \mathrm{Out}(\pi)$ has index $\ge d$.
- Consequently, no subgroup $G < \mathrm{Out}(\pi)$ of index $\le d$ lifts to $\mathrm{Diff}(N)$.

An exotic smooth structure

$M = \mathbb{H}^n/\pi$; N exotic smooth structure

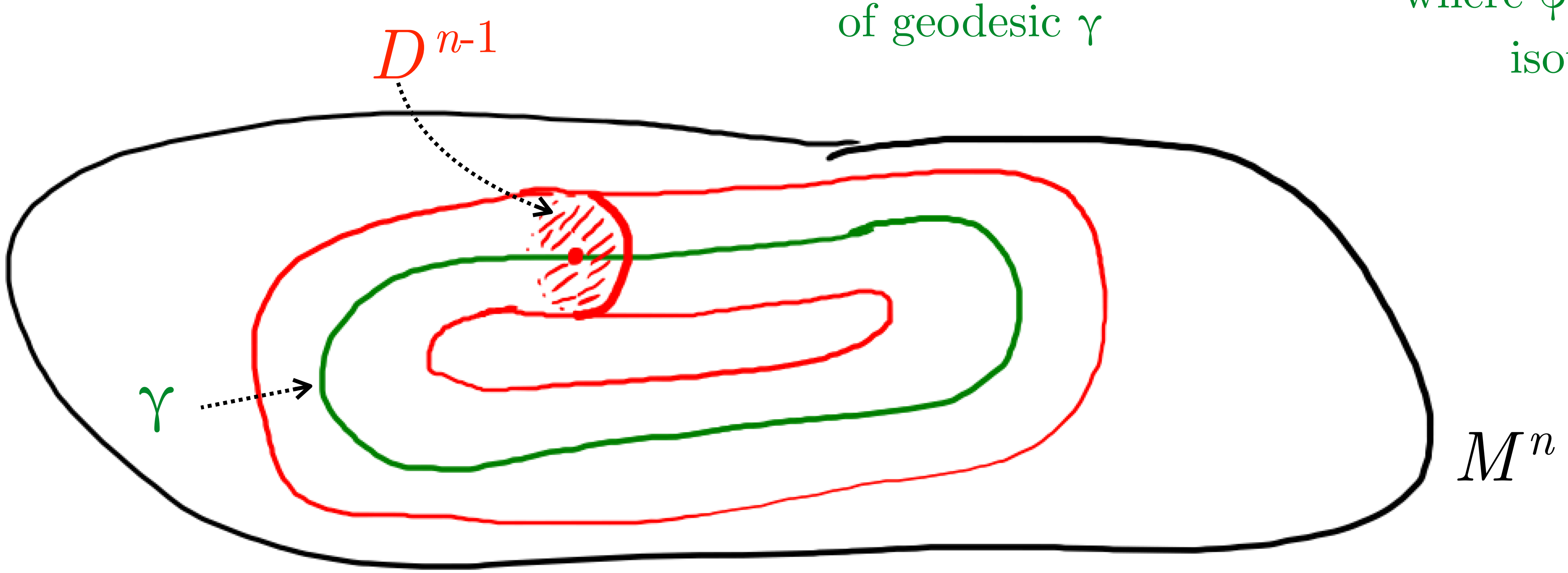
$s(N) := \max \{ \, |G| : G < \mathrm{Diff}(N) \text{ finite} \}$

Theorem (Bustamante-T). For each $d \geq 2$, there exists M, N so that $s(N) \leq \frac{1}{d}|\mathrm{Out}(\pi)|$.

Defining N: $N_{\gamma,\varphi} = M \setminus S^1 \times D^{n-1} \cup S^1 \times D^{n-1}$

neighborhood of geodesic γ

glue by $1 \times \varphi \in \mathrm{Diff}(S^1 \times S^{n-2})$ where $\varphi \in \mathrm{Diff}(S^{n-2})$ not isotopic to id.

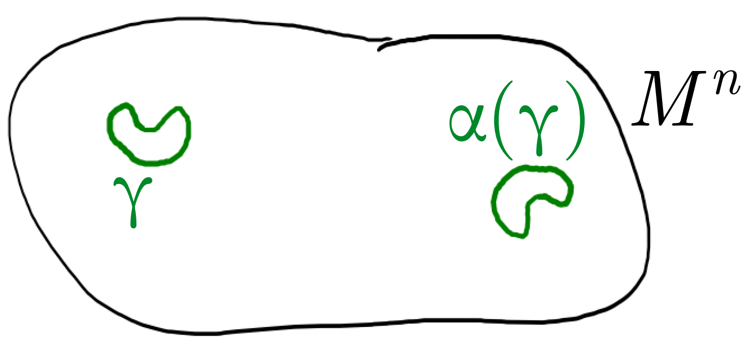


About the Proof

$M = \mathbb{H}^n/\pi$; $N_{\gamma,\varphi} = M \setminus S^1 \times D^{n\text{-}1} \cup S^1 \times D^{n\text{-}1}$

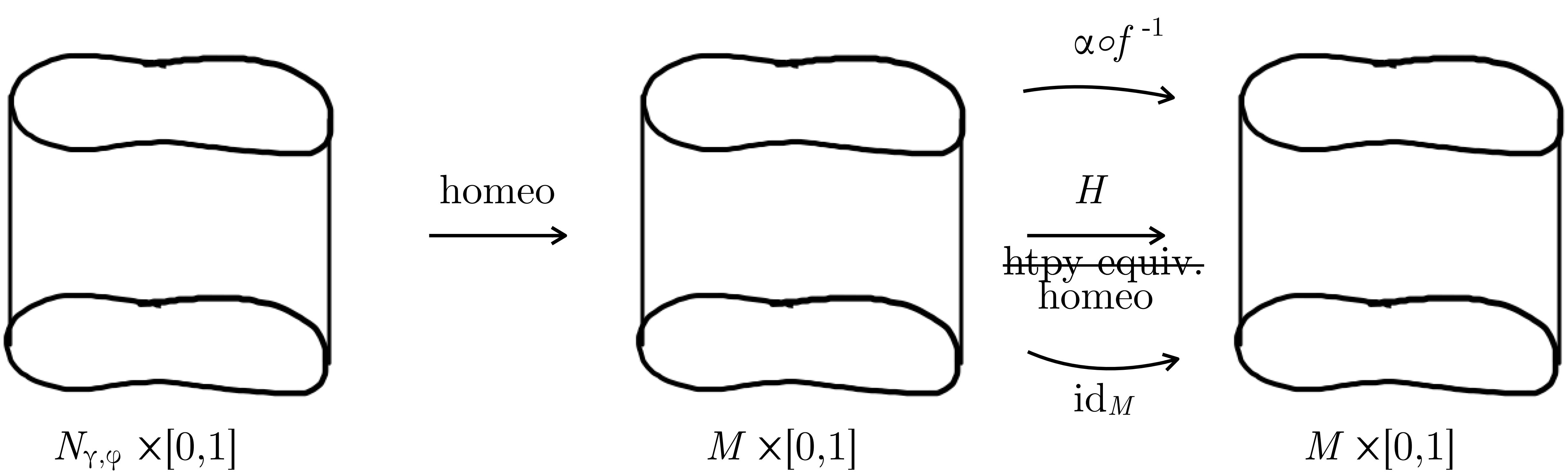
<u>Want</u>: $\alpha \not\in \mathrm{Im}[\mathrm{Diff}(N_{\gamma,\varphi}) \to \mathrm{Out}(\pi)]$, $\mathrm{order}(\alpha)=d$.

<u>Key observation</u>: If $\exists\, f \in \mathrm{Diff}(N_{\gamma,\varphi})$ inducing $\alpha \in \mathrm{Out}(\pi) \cong \mathrm{Isom}(M)$ then $N_{\gamma,\varphi}$ and $N_{\alpha(\gamma),\varphi}$ are *concordant* smooth structures.



$$M \simeq N_{\gamma,\varphi} \xrightarrow{f^{-1}} N_{\gamma,\varphi} \xrightarrow{\alpha} N_{\alpha(\gamma),\varphi} \simeq M$$

homeomorphism, homotopic to id_M



Question

Does there exist M with $|\mathrm{Isom}(M)| \gg 1$ and N exotic smooth structure so that $\mathrm{s}(N)=1$?

Equivalently, $\mathrm{Diff}(N)$ has no nontrivial finite order element.

Equivalently, $\mathrm{Diff}(N) \to \mathrm{Out}(\pi)$ is trivial.

Thank you.