From Artin Monoids to Artin Groups

Ruth Charney

today 2 favorite types of groups:

Artin groups $\rightsquigarrow$ Coxeter groups

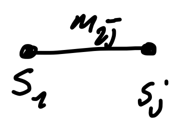
I. Definitions and Examples

Artin group: $A = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \rangle$

$m_{ij} \in \{2, 3, \dots, \infty\}$

Encode this presentation in a graph Γ:

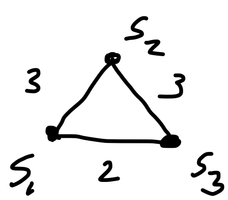
Γ vertices $\{s_1, \dots s_n\}$

edge 

$m_{ij} \neq \infty$

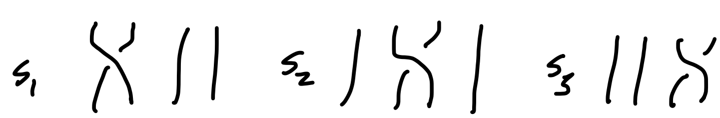
denote group by A_Γ

example: Γ



$$A_\Gamma = \langle s_1, s_2, s_3 \mid s_1 s_3 = s_3 s_1,\ s_1 s_2 s_1 = s_2 s_1 s_2,\ s_2 s_3 s_2 = s_3 s_2 s_3 \rangle$$

= braid group on 4-strands



if we add relations $s_i^2 = 1$ $\forall i$ then we get Coxeter group W_Γ

for Γ above W_Γ = symmetric group on 4 letters

These groups fall into 2 classes

W_Γ finite $\Updownarrow$ A_Γ finite type (A.K.A. spherical type) — well-understood

W_Γ infinite $\Updownarrow$ A_Γ infinite type — totally mysterious

Why?

The Artin monoid A_Γ^+ is the monoid defined by the same presentation (i.e. don't have inverses)

A_Γ^+ has nice combinatorial properties for all Γ

- canonical forms
- solution to word problem

if A_Γ is of finite type, then A_Γ^+ contains a special element Δ called Garside element such that

longest word in W_Γ lift to A_Γ

any $g \in A_\Gamma$ can be written as

$$g = a\Delta^{-n}$$

some $a \in A_\Gamma^+$ $n \in \mathbb{Z}$

so turn word problems in A_Γ to A_Γ^+

For A_Γ infinite type, there is no analogue of the Garside element

How do we get from A_Γ^+ to A_Γ in this case?

II. Joint work with R. Boyd, R. Morris-Wright, S. Rees

Geometric approach to Artin groups

Coxeter groups can be realized as reflection group W_Γ acting on $\mathbb{C}^n$

Each reflection $r \in W_\Gamma$ fixes a hyperplane $H_r \subset \mathbb{C}^n$ and W_Γ acts freely on $\mathcal{H} = \mathbb{C}^n - \bigcup_r H_r$

Th^m (van der Lek '83):

$$\pi_1(\mathcal{H}_\Gamma / W_\Gamma) = A_\Gamma$$

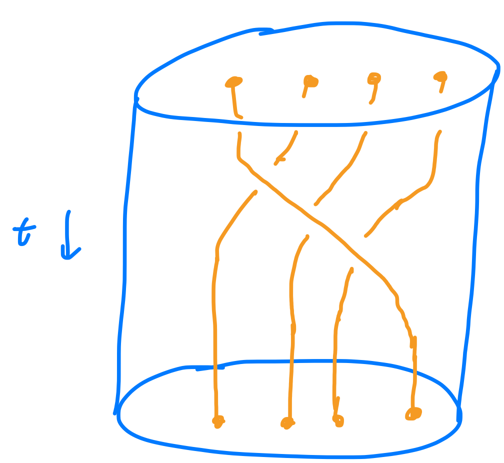
Example:

A_Γ = braid group

W_Γ = symmetric group $\curvearrowright \mathbb{C}^n$ on n-letters

hyperplanes = pts with 2 coords same

$\mathcal{H}_\Gamma$ = configuration space of n distinct pts in $\mathbb{C}^n$



a loop in this space is a braid

$K(\pi,1)$-conjecture: $\mathcal{H}_\Gamma/W_\Gamma$ is a $K(\pi,1)$-space for A_Γ

equivalently $\widetilde{\mathcal{H}_\Gamma/W_\Gamma}$ is contractible

universal cover

Conjecture holds for:

* all finite type A_Γ (Deligne '72)
* some infinite type
  FC-type, 2-dim'l, some 3-dim'l, ...
  (C-Davis '95, C '03)
* Euclidean type (Paolini-Salvetti '20)

$*$ main idea

proof involves 2 steps

* Construct a simplicial complex $\mathcal{D}_\Gamma$
  s.t. $\mathcal{D}_\Gamma \simeq \widetilde{\mathcal{H}_\Gamma}/W_\Gamma$
* Prove $\mathcal{D}_\Gamma \simeq *$

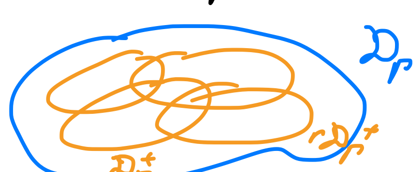
New work:

Construct an analogous symplicial complex $\mathcal{D}_\Gamma^+$ for A_Γ^+ and prove

Th$^{\underline{m}}$ (Boyd, C, Morris-Wright)

$\mathcal{D}_\Gamma^+$ is contractible for all Γ
and it embeds in $\mathcal{D}_\Gamma$

A_Γ acts on $\mathcal{D}_\Gamma$ and the A_Γ translates of $\mathcal{D}_\Gamma^+$ covers all of $\mathcal{D}_\Gamma$



need to understand intersections of these translates

$\Updownarrow$

need to understand different ways of expressing

$g \in A_\Gamma$ as a product

$$g = a_1 a_2^{-1} a_3 a_4^{-1} \cdots \qquad a_i \in A_\Gamma^+$$

$\Updownarrow$

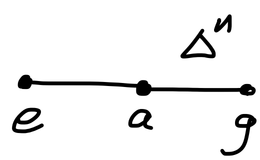
understand Cayley graph

$$Cay^+(A_\Gamma) = Cay(A_\Gamma, A_\Gamma^+)$$

<u>example:</u>

If A_Γ is finite type we can write any $g \in A_\Gamma$

as $g = a\Delta^{-n}$ $\quad a \in A_\Gamma^+ \quad \Delta^n \in A_\Gamma^{-1}$



$$\mathrm{diam}(Cay^+) = 2$$

<u>Questions (A_Γ infinite type):</u>

1) <u>Conj:</u> $Cay^+(A_\Gamma)$ has infinite diam

Thm (B-C-MW-R)

conjecture holds if

- some $m_{ij} = \infty$ or
- Γ contains a triangle with all $m_{ij} \geq 3$

2) Try to solve word problem in $Cay^+(A_\Gamma)$

Dehornoy proposed 2 algorithms

for this

- convergence (only works for FC-type)
- semiconvergence (conjectures this holds for all A_Γ)

<u>Thm</u>: could use this to prove

$\mathcal{D}_\Gamma$ contractible

<u>Conj</u>: semiconvergence $\Rightarrow$ $\mathcal{D}_\Gamma$ contractible