Links and Rational Homology 4-balls

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Question: Which rational homology 3-spheres ($\mathbb{Q}S^3$'s) bound rational homology 4-balls ($\mathbb{Q}B^4$'s)

History:

- Fintushel-Stern (80's): Brieskorn spheres bounding $\mathbb{Q}B^4$'s
  Casson-Harer
- Lisca ('07): Classified lens spaces and sums of lens spaces bounding $\mathbb{Q}B^4$'s
- Lecuona ('09): Small Seifert fiber spaces bounding $\mathbb{Q}B^4$'s
- \+ others

K is slice if it bounds a smooth properly embedded disk in B^4

A construction: If $K \subset S^3$ is a slice knot, then the double cover of S^3 branched along K, $\Sigma_2(S^3, K)$ bounds a $\mathbb{Q}B^4$, namely $\Sigma_2(B^4, D)$, where D is a slice disk for K.

Question: How can we generalize this construction to links?

Fact: $\Sigma_2(S^3, L)$ is a $\mathbb{Q}S^3 \Leftrightarrow \det L \neq 0$

- Slice Links?

An n-component link L is slice if it bounds a smooth properly embedded $\bigsqcup_n D^2$ in B^4

For $n \geq 2$, $\det L = 0$ (in fact, $\Sigma_2(S^3, L)$ is a $\mathbb{Q}(\#^{n-1} S^1 \times S^2)$)

$\Rightarrow \Sigma_2(S^3, L)$ cannot bound a $\mathbb{Q}B^4$

- χ-slice links (Donald-Owens '12)

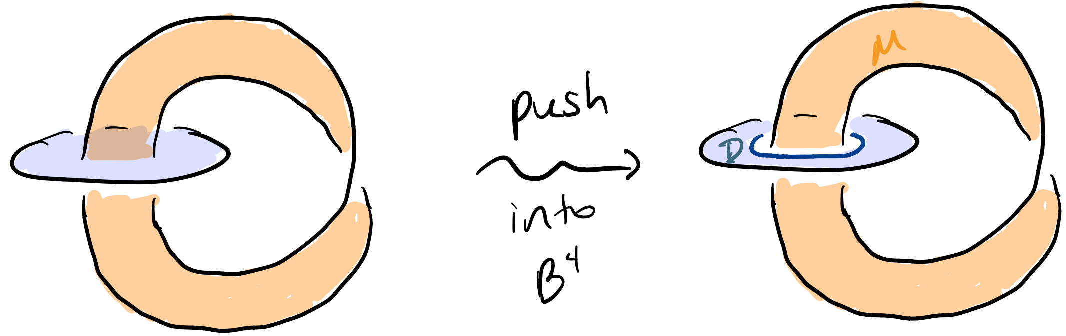
  A link L is χ-slice if L bounds a smooth properly embedded surface F in B^4 with no closed components and $\chi(F) = 1$.

Note:
- F can be disconnected and unorientable
- If L is a knot, then L is χ-slice $\Leftrightarrow$ L is slice

  (since the only surface with one boundary component and $\chi = 1$ is D^2)

Prop (Donald-Owens): If $\det L \neq 0$ and L is χ-slice, then $\Sigma_2(S^3, L)$ is a $\mathbb{Q}S^3$ that bounds a $\mathbb{Q}B^4$, namely $\Sigma_2(B^4, F)$, where F is a χ-slice surface for L.

Exs: $T(2,4)$



$F = M \cup D \subset B^4$

$\chi(F) = 1$

- Casson-Harer, Lecuona give examples of χ-slice Montesinos links

- Lisca classifies χ-slice 2-bridge links
  (in his classification of lens spaces bounding $\mathbb{Q}B^4$'s)

Slice vs. χ-slice

Let L be an n-component link

- If $\det L \neq 0$ and $F \subset B^4$ with $\partial F = L$, then $\chi(F) \leq 1$

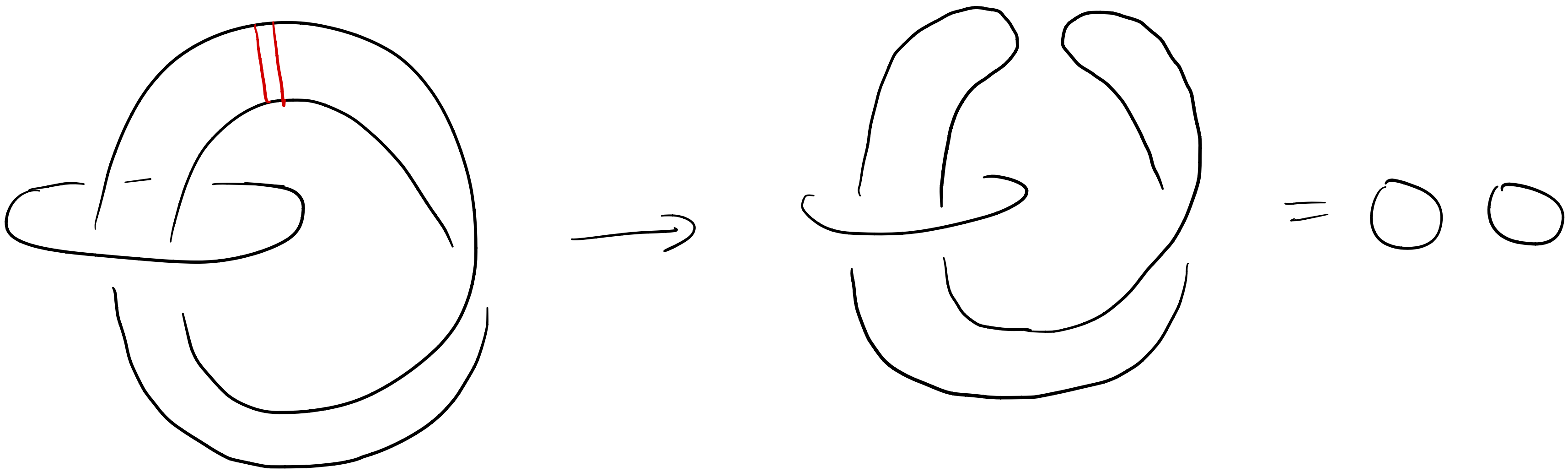
  (and $\chi(F) = 1 \Rightarrow L$ is χ-slice)

- If $\det L = 0$ and $F \subset B^4$ with $\partial F = L$, then $\chi(F) \leq n$

  (and $\chi(F) = n \Rightarrow L$ is slice)

Construction

To show a link is χ-slice, can use band moves



If n band moves yield $\underbrace{\bigcirc \;- - -\; \bigcirc}_{n+1}$

then L is χ-slice

Obstruction

- Prove $\Sigma_2(S^3, L)$ does not bound a $\mathbb{Q}B^4$
  - <u>$\det L$</u>

    If $\Sigma_2(S^3,L)$ bounds a $\mathbb{Q}B^4$, then $\det L = |H_1(\Sigma_2(S^3,L))|$ is a square.
  - <u>Donaldson's Diagonalization Theorem</u>

    If Z is a smooth, closed, oriented, negative-definite 4-manifold, then its intersection form Q_Z is diagonalizable

    $\Rightarrow$ If $\Sigma_2(S^3,L)$ bounds a $\mathbb{Q}B^4$, B and a negative-definite 4-manifold X, then $\exists$ a lattice embedding

    $$(H_2(X \cup -B), Q_{X\cup -B}) \longrightarrow (\mathbb{Z}^{\operatorname{rank}(H_2(X\cup -B))}, -I)$$
  - <u>Heegaard Floer homology d-invariants</u>

    If $\Sigma_2(S^3,L)$ bounds a $\mathbb{Q}B^4$, then there are at least $\sqrt{|H_1(\Sigma_2(S^3,L))|}$ vanishing d-invariants.

- Partial χ-sliceness obstructions

  - Signatures (Donald-Owens)

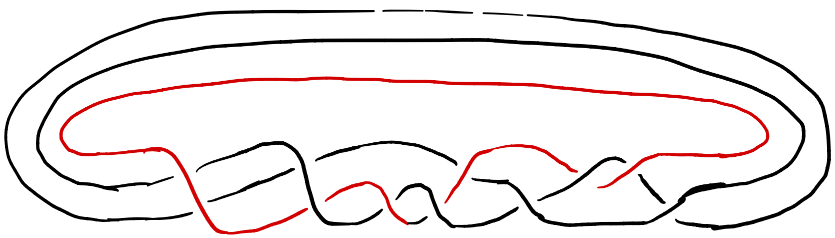
    If L bounds an <u>oriented</u> χ-slice surface, then $\sigma(L)=0$

  - Fox-Milnor Condition (Florens)

    If L bounds an <u>orientable</u> χ-slice surface, then the multivariable Alexander polynomial satisfies the Fox-Milnor condition

3-braid links

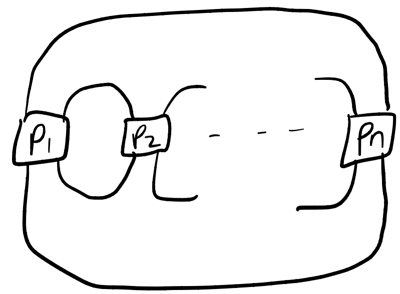
(w/ Vitaly Brejevs)

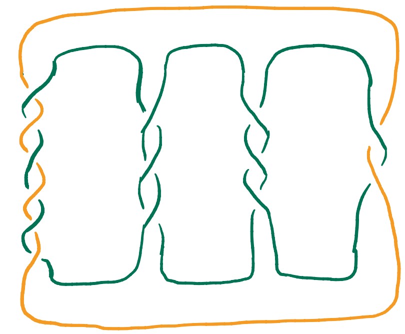


- classify nonalternating quasialternating 3-braid links that are χ-slice

Pretzel Links

(w/ Sophia Fanelle, Ben Huenemann, Evan Huang, Weizhe Shen, Hannah Turner)

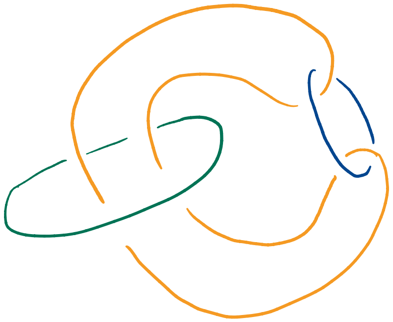




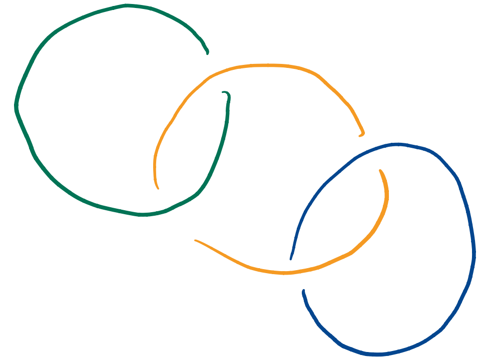
(4-stranded pretzel link)

- classify positive and negative pretzel links that are χ-slice
- give partial classification of 3- and 4-stranded pretzel links that are χ-slice

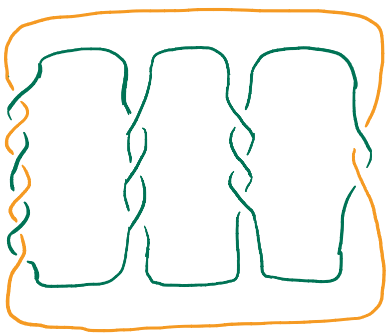
These results extend known results regarding the sliceness of 3-braid knots and pretzel knots.



(Borromean Rings)



Thanks!

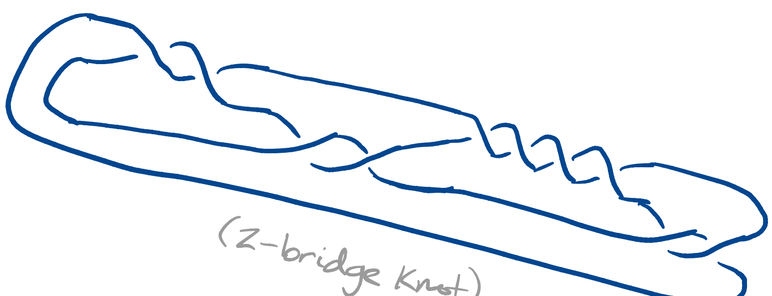


(4-stranded pretzel link)

Challenge!

All links on this page are x-slice.

Can you find band moves to prove it?



(2-bridge knot)