Stably Weinstein Liouville domains

Austin Christian
joint with J. Breen

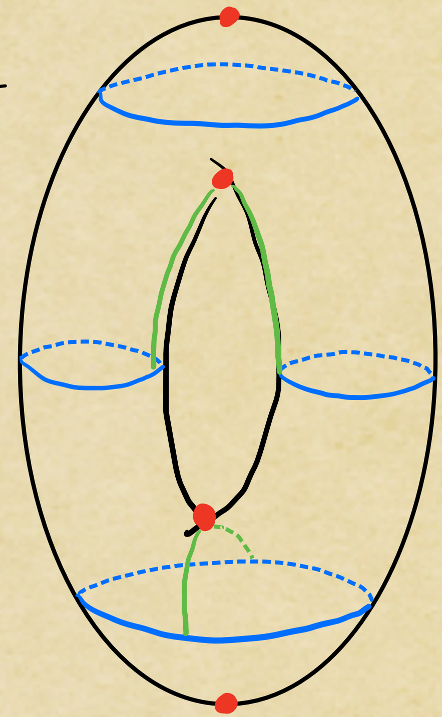
Tech Topology 2023

Outline

1. Morse theory for geometric objects
   (What are Weinstein domains?)
2. A topological obstruction
   (Liouville $\not\Rightarrow$ Weinstein)
3. Explicit manipulations
4. Stably Weinstein Liouville domains

§1 Morse theory for geometric objects

Much of the importance of Morse theory lies in its correspondence with handle calculus.

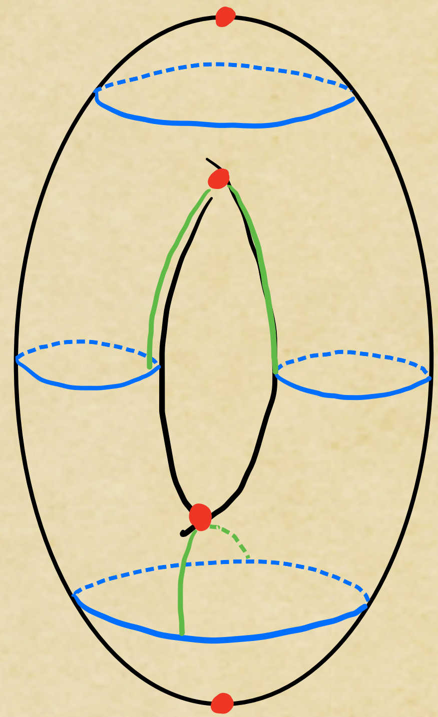


{Morse functions} ⟷ {handle decompositions}

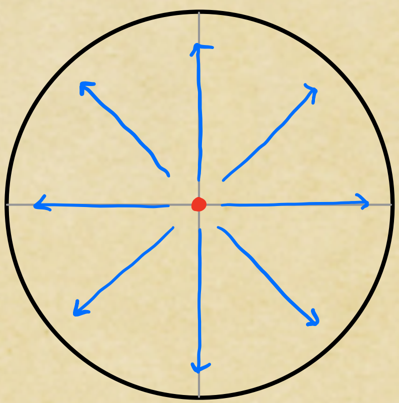
{h'topies of functions} ⟷ {creation/cancellation/isotopy of handles}

§1 Morse theory for geometric objects

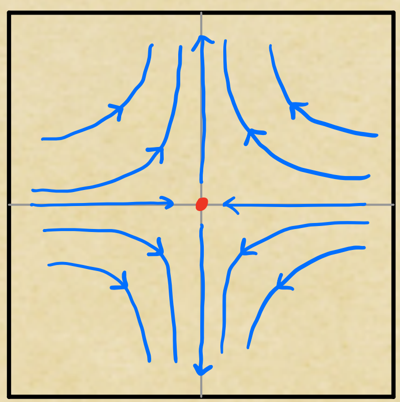
This correspondence passes thru vector fields, and these will be where we focus our attention.



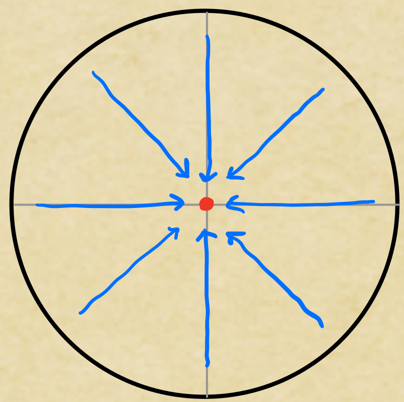
index 0 C.P.



index 1 C.P.

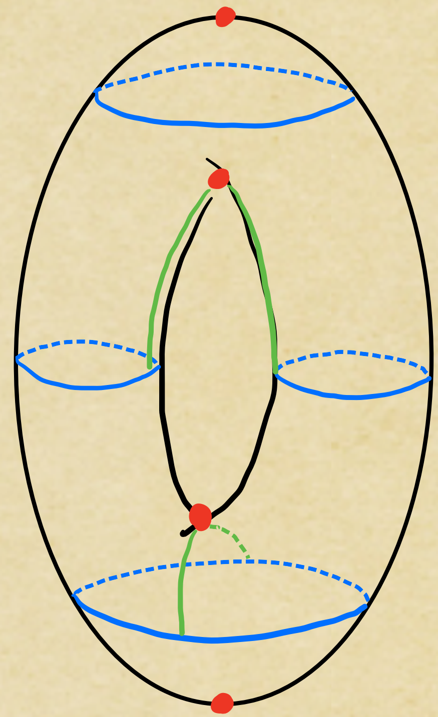


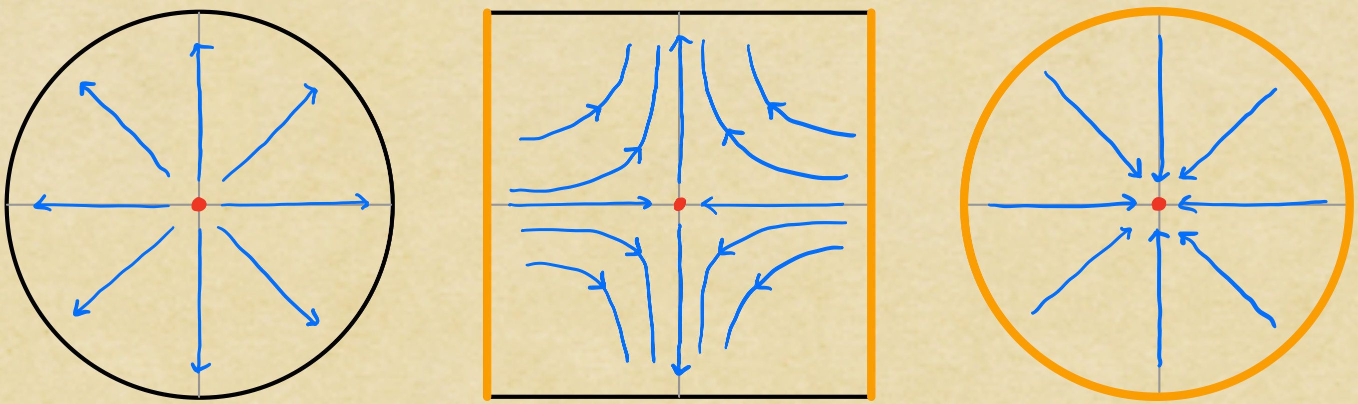
index 2 C.P.



§1 Morse theory for geometric objects

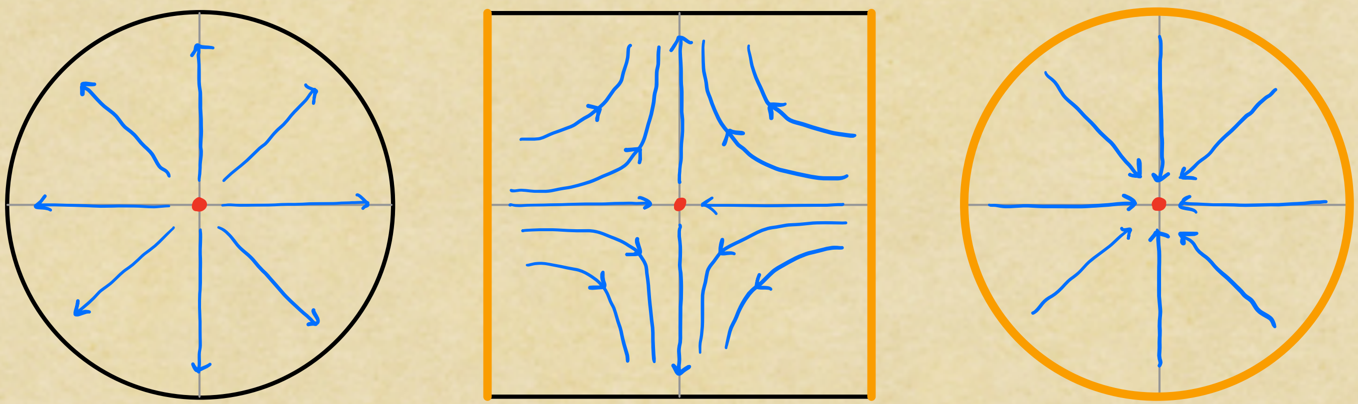
This correspondence passes thru vector fields, and these will be where we focus our attention.





§1 Morse theory for geometric objects

index 0 C.P. | index 1 C.P. | index 2 C.P.



0-handle | 1-handle | 2-handle

Note Obtaining a handle decomposition will require this vector field to be gradient like for a Morse function.

§1 Morse theory for geometric objects

A key feature of the relevant vector fields is that their flows

$$\Phi_t : M \longrightarrow M$$

are diffeomorphisms.

§1 Morse theory for geometric objects

A key feature of the relevant vector fields is that their flows

$$\Phi_t : M \longrightarrow M$$

are diffeomorphisms.

If M carries a geometrical structure, we want Φ_t to be an isomorphism in the relevant category.

Ex. If M were Riemannian, we'd look for a gradient-like vector field whose flows $\Phi_t : M \to M$ are isometries.

§1 Morse theory for geometric objects

Today we're interested in contact & symplectic geometries.

Def. A contact structure on M^{2n+1} is a codimension 1 distribution $\xi \subset TM$ which can be written $\xi = \ker \alpha$, with $\alpha \in \Omega^1(M)$ satisfying $\alpha \wedge (d\alpha)^n > 0$.

Def. A symplectic structure on M^{2n} is a closed 2-form $\omega \in \Omega^2(M)$ satisfying $\omega^n > 0$.

§1 Morse theory for geometric objects

Today we're interested in contact & symplectic geometries.

Def. A contact structure on M^{2n+1} is a codimension 1 distribution $\xi \subset TM$ which can be written $\xi = \ker \alpha$, with $\alpha \in \Omega^1(M)$ satisfying $\alpha \wedge (d\alpha)^n > 0$.

Def. A symplectic structure on M^{2n} is a closed 2-form $\omega \in \Omega^2(M)$ satisfying $\omega^n > 0$.

Hope We want a gradient-like vector field s.t.

$$\mathcal{L}_X \alpha = g\alpha \quad \text{OR} \quad \mathcal{L}_X \omega = \omega.$$

§1 Morse theory for geometric objects

Giroux: All contact manifolds admit contact vector fields (i.e., vector fields $X \in \mathfrak{X}(M)$ s.t. $\mathcal{L}_X \alpha = g\alpha$) which are gradient-like.

§1 Morse theory for geometric objects

Giroux: All contact manifolds admit contact vector fields (i.e., vector fields $X \in \mathfrak{X}(M)$ s.t. $\mathcal{L}_X \alpha = g\alpha$) which are gradient-like.

On a closed symplectic manifold (M^{2n}, ω), our hopes are dashed: $\mathcal{L}_X \omega = 0 \Rightarrow \mathcal{L}_X \omega^n = 0 \Rightarrow$ flow of X is volume-preserving $\Rightarrow$ no C.P.s of index 0 or $2n$.

§1 Morse theory for geometric objects

Giroux: All contact manifolds admit contact vector fields (i.e., vector fields $X \in \mathfrak{X}(M)$ s.t. $\mathcal{L}_X \alpha = g\alpha$) which are gradient-like.

On a closed symplectic manifold (M^{2n}, ω), our hopes are dashed: $\mathcal{L}_X \omega = 0 \Rightarrow \mathcal{L}_X \omega^n = 0 \Rightarrow$ flow of X is volume-preserving $\Rightarrow$ no C.P.s of index 0 or $2n$.

We have a replacement condition which allows conformal changes: $\mathcal{L}_X \omega = \omega$. This condition is modeled on the standard geometry of T^*Q.

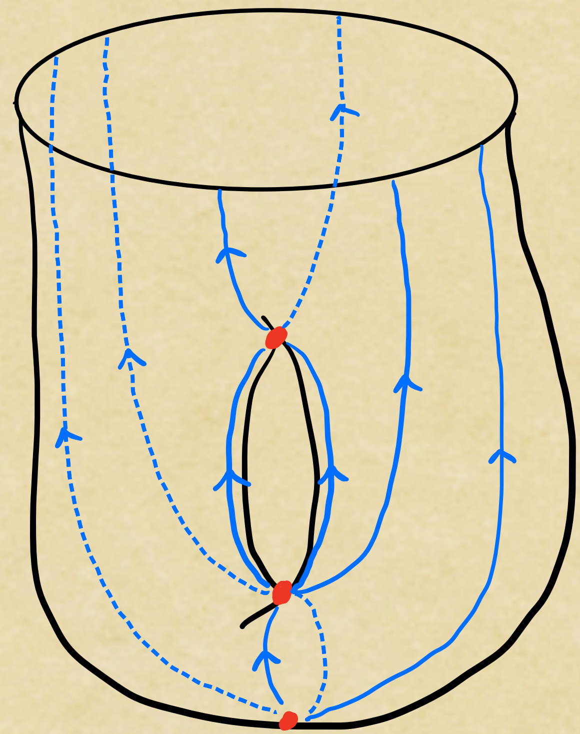
§1 Morse theory for geometric objects

Def. A Liouville domain is a pair (W, λ), where

- $(W, d\lambda)$ is a compact symplectic manifold;
- the unique vector field X_λ satisfying

$$\iota_{X_\lambda}(d\lambda) = \lambda \quad \left(\Longrightarrow \mathcal{L}_{X_\lambda}(d\lambda) = d\lambda\right)$$

is outwardly transverse to ∂W.



§1 Morse theory for geometric objects

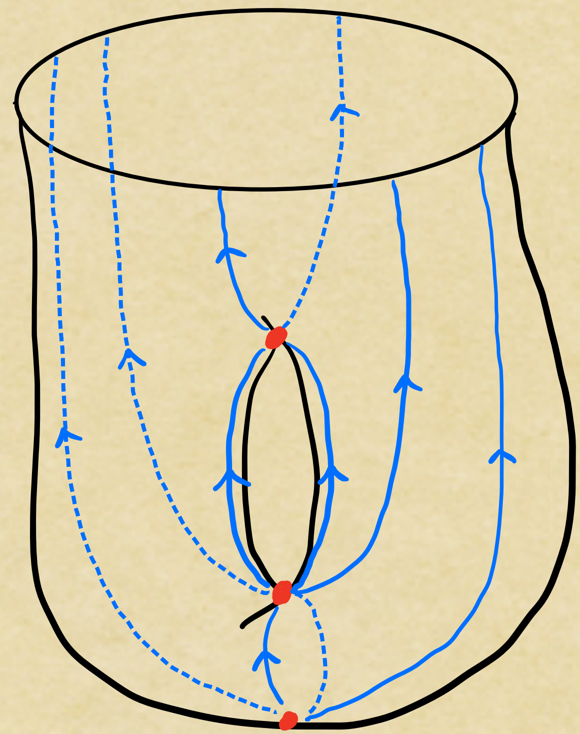
Def. A Liouville domain is a pair (W, λ), where

- $(W, d\lambda)$ is a compact symplectic manifold;
- the unique vector field X_λ satisfying

$$\iota_{X_\lambda}(d\lambda) = \lambda \quad \left(\Longrightarrow \mathcal{L}_{X_\lambda}(d\lambda) = d\lambda\right)$$

  is outwardly transverse to ∂W.

A Weinstein domain is a Liouville domain (W, λ) whose Liouville vector field X_λ is gradient-like for some Morse function.



§1 Morse theory for geometric objects

Liouville: (W,λ) s.t. $(W, d\lambda)$ is sympl. & $X_\lambda \pitchfork \partial W$

$\iota_{X_\lambda}(d\lambda) = \lambda$

Weinstein: Also need X_λ to be gradient-like for some Morse function.

Remark. In either case, Stokes' theorem requires $\partial W \neq \emptyset$. If $\omega = d\lambda$, then

$$\int_{\partial W} (\iota_X \omega) \wedge \omega^{n-1} = \int_W (\mathcal{L}_X \omega) \wedge \omega^{n-1} = \int_W \omega^n > 0.$$

§1 Morse theory for geometric objects

Liouville: (W, λ) s.t. $(W, d\lambda)$ is sympl. & $X_\lambda \pitchfork \partial W$ ← $\iota_{X_\lambda}(d\lambda) = \lambda$

Weinstein: Also need X_λ to be gradient-like for some Morse function.

Remark. In either case, Stokes' theorem requires $\partial W \neq \emptyset$. If $\omega = d\lambda$, then

$$\int_{\partial W} (\iota_X \omega) \wedge \omega^{n-1} = \int_W (\mathcal{L}_X \omega) \wedge \omega^{n-1} = \int_W \omega^n > 0.$$

Q. How common are Weinstein domains among Liouville domains?

§2 A topological obstruction

Many compact mflds are precluded from carrying a Weinstein structure by their topology.

Prop. If (W^{2n}, λ) is a Weinstein domain and $\phi : W \to \mathbb{R}$ is a Morse function for which X_λ is gradient-like, then every critical point of ϕ has index $\leq n$.

§2 A topological obstruction

Many compact mflds are precluded from carrying a Weinstein structure by their topology.

Prop. If (W^{2n}, λ) is a Weinstein domain and $\phi : W \to \mathbb{R}$ is a Morse function for which X_λ is gradient-like, then every critical point of ϕ has index $\leq n$.

Cor. In dim ≥ 4, Weinstein domains must have connected boundary.

c.f. complex mfld with pseudo-convex bdry

§2 A topological obstruction

Many compact mflds are precluded from carrying a Weinstein structure by their topology.

Indeed, there are* Liouville domains* whose topology precludes a Weinstein structure.

Ex (McDuff '91) $\exists$ L'ville structure on

$$DT^*\Sigma_g - N,$$

for any $g>1$, where $N =$ nbhd of zero section.

disconnected bdry $\Rightarrow \nexists$ Weinstein structure

§2 A topological obstruction

Many examples now exist, in all even dimensions.

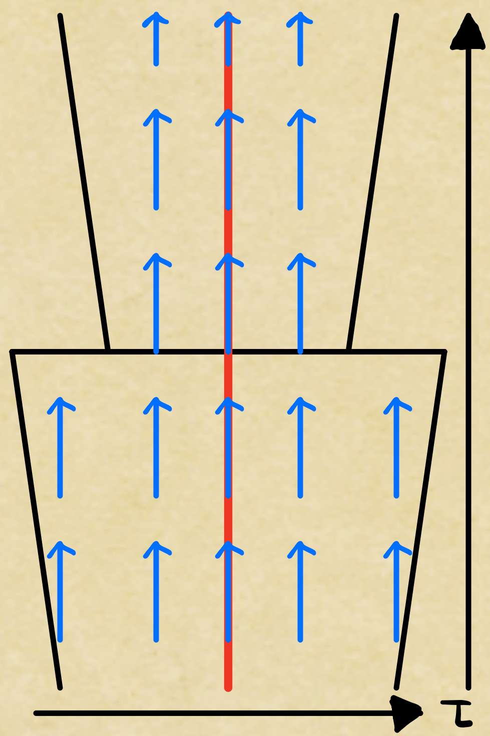
Ex (Mitsumatsu '95, Geiges '95)

Pick $A \in SL(2,\mathbb{Z})$ with eigenvalues $0 < \lambda_2 < 1 < \lambda_1$.

$\exists$ contact form α on $M = [-1,1]_\tau \times T^2$ for which

$$\Phi_A : M \longrightarrow M$$
$$\left(\tau, \begin{pmatrix} x \\ y \end{pmatrix}\right) \mapsto \left(\frac{\lambda_2}{\lambda_1}, A\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

is a contacto. onto its image.



§2 A topological obstruction

Many examples now exist, in all even dimensions.

__Ex__ (Mitsumatsu '95, Geiges '95)

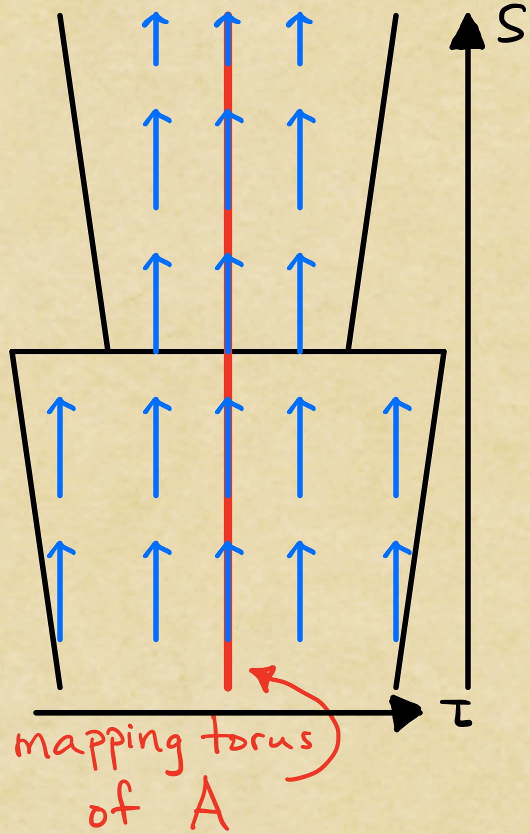
Pick $A \in SL(2,\mathbb{Z})$ with eigenvalues $0 < \lambda_2 < 1 < \lambda_1$.

$\exists$ contact form α on $M = [-1,1]_\tau \times T^2$ for which

$$\Phi_A : M \longrightarrow M$$
$$\left(\tau, \begin{pmatrix} x \\ y \end{pmatrix}\right) \mapsto \left(\frac{\lambda_2}{\lambda_1}, A\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

is a contacto. onto its image.

The partial mapping torus $(W_A, e^s\alpha)$ is a Liouville domain.



§2 A topological obstruction

Many examples now exist, in all even dimensions.

Ex (Mitsumatsu '95, Geiges '95)

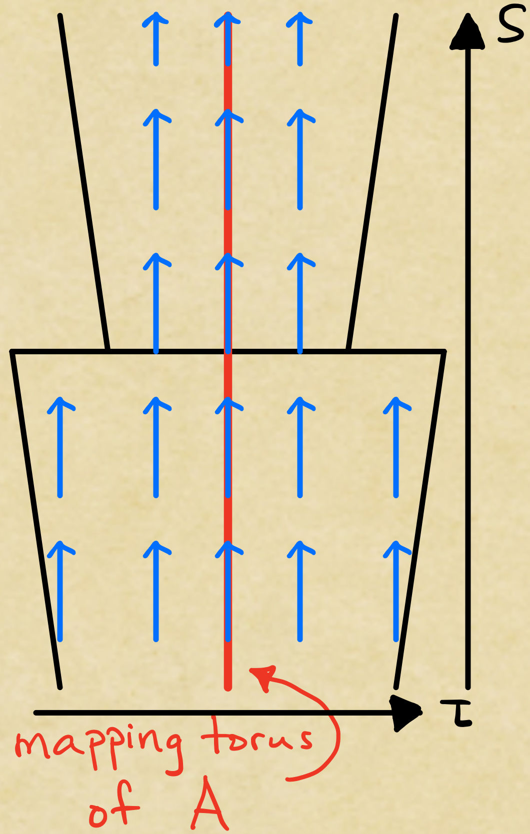
Pick $A \in SL(2,\mathbb{Z})$ with eigenvalues $0 < \lambda_2 < 1 < \lambda_1$.

$\exists$ contact form α on $M = [-1,1]_\tau \times T^2$ for which

$$\Phi_A : M \longrightarrow M$$
$$\left(\tau, \begin{pmatrix} x \\ y \end{pmatrix}\right) \mapsto \left(\frac{\lambda_2}{\lambda_1}, A\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

is a contacto. onto its image.

The partial mapping torus $(W_A, e^s\alpha)$ is a Liouville domain.



Huang '20: Analogous construction in all dimensions.

§2 A topological obstruction

Q Is the index bound the only obstruction to making a Liouville domain Weinstein?

Concretely, if (W^{2n}, λ_0) is a Liouville domain and W is built from handles of index $\leq n$, $\exists$ L'ville h'topy λ_t s.t. (W, λ_1) is Weinstein?

§2 A topological obstruction

Q Is the index bound the only obstruction to making a Liouville domain Weinstein?

Concretely, if (W^{2n}, λ_0) is a Liouville domain and W is built from handles of index $\leq n$, $\exists$ L'ville h'topy λ_t s.t. (W, λ_1) is Weinstein?

Simplification: Are all Liouville domains stably Weinstein?

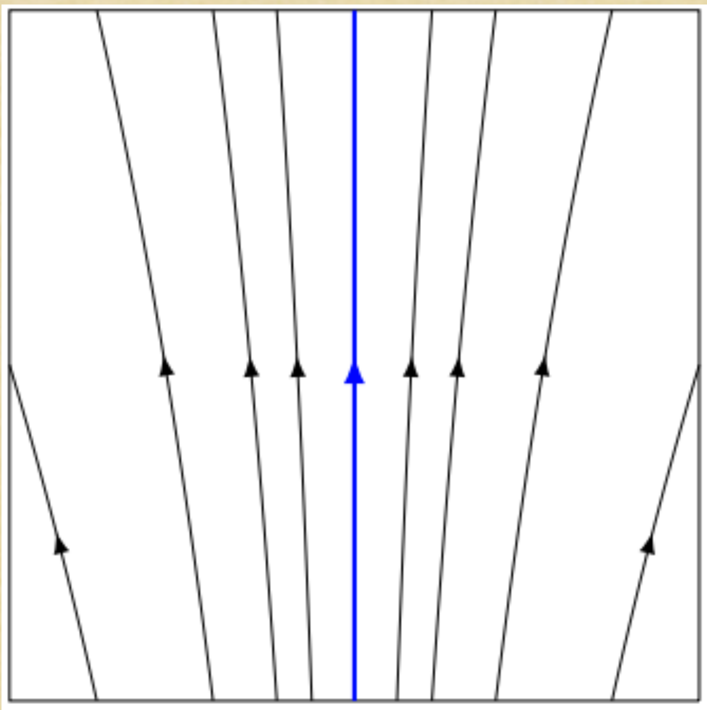
stabilization: $(W, \lambda) \rightsquigarrow (W \times D^2, \lambda + \lambda_{std})$

dimension increases, handle indices do not

§3 Explicit manipulations

Are all Liouville domains stably Weinstein?

We approach this as a dynamical problem, and directly manipulate the Liouville v.f. X_λ of (W, λ).

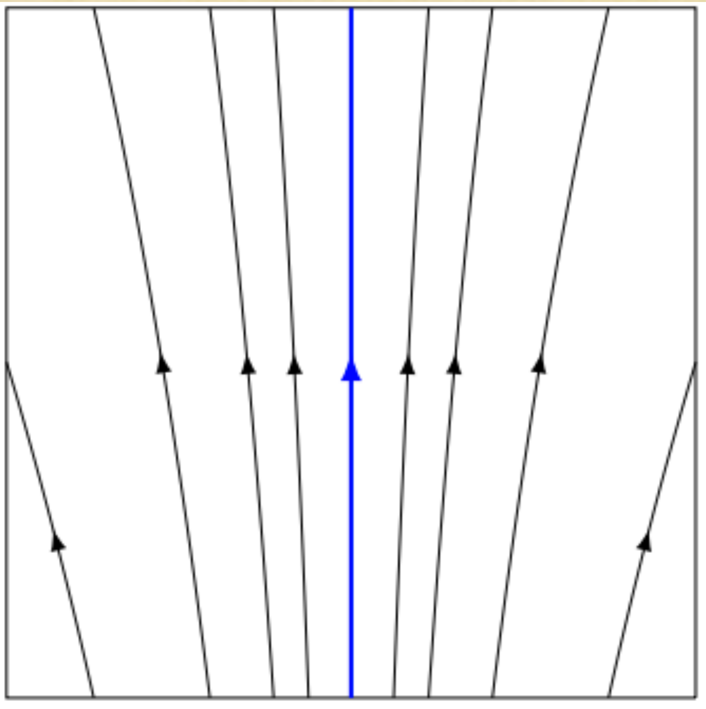


$S^1 \times I$

§ 3 Explicit manipulations

Are all Liouville domains stably Weinstein?

We approach this as a dynamical problem, and directly manipulate the Liouville v.f. X_λ of (W, λ).



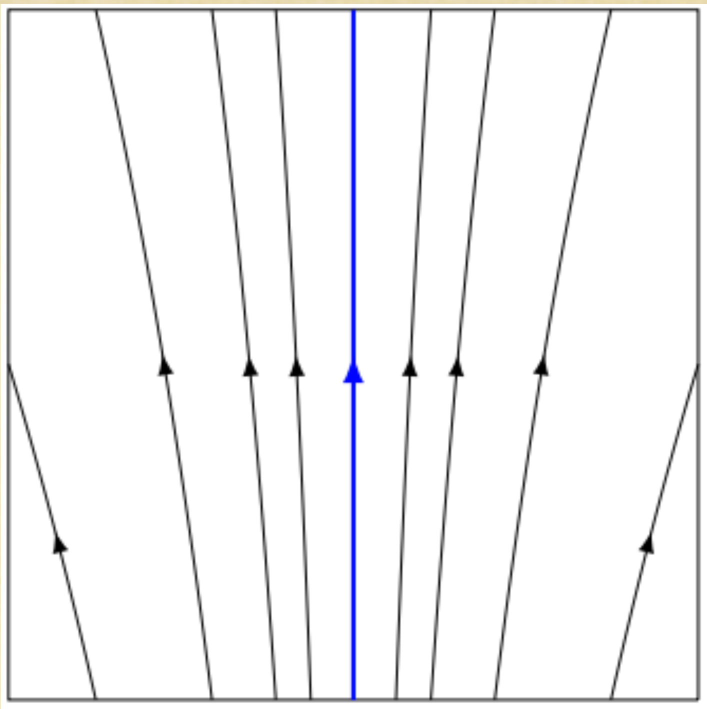
$S^1 \times I$

By applying local, compactly supported Liouville homotopies, we hope to interrupt cyclic flowlines of X_λ, and trap wandering flowlines in backward time.

§3 Explicit manipulations

Prop. A Liouville domain (W, λ) is Weinstein iff

(1) Every flowline of X_λ converges to a C.P. in backward time.



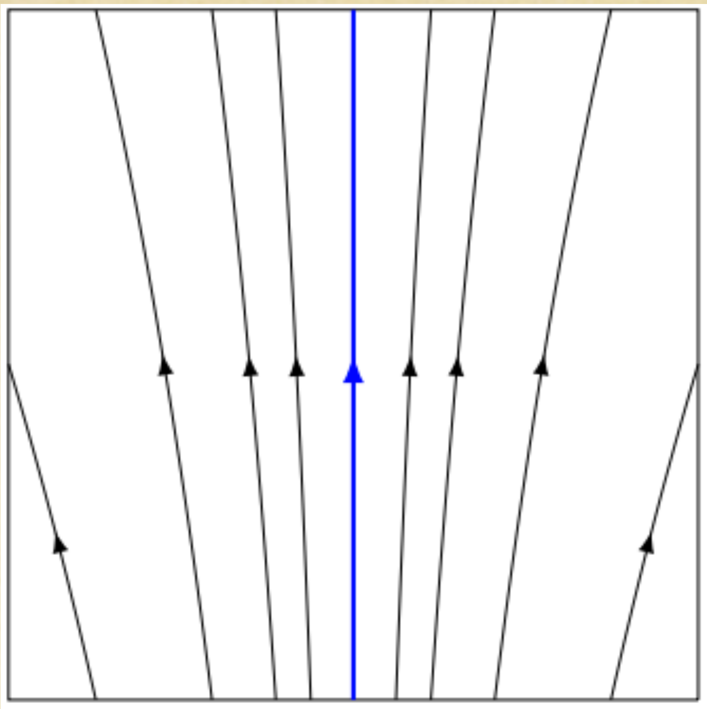
$S^1 \times I$

§3 Explicit manipulations

Prop. A Liouville domain (W, λ) is Weinstein iff

(1) Every flowline of X_λ converges to a C.P. in backward time.

(2) The flowlines of X_λ include no loops, broken or otherwise.

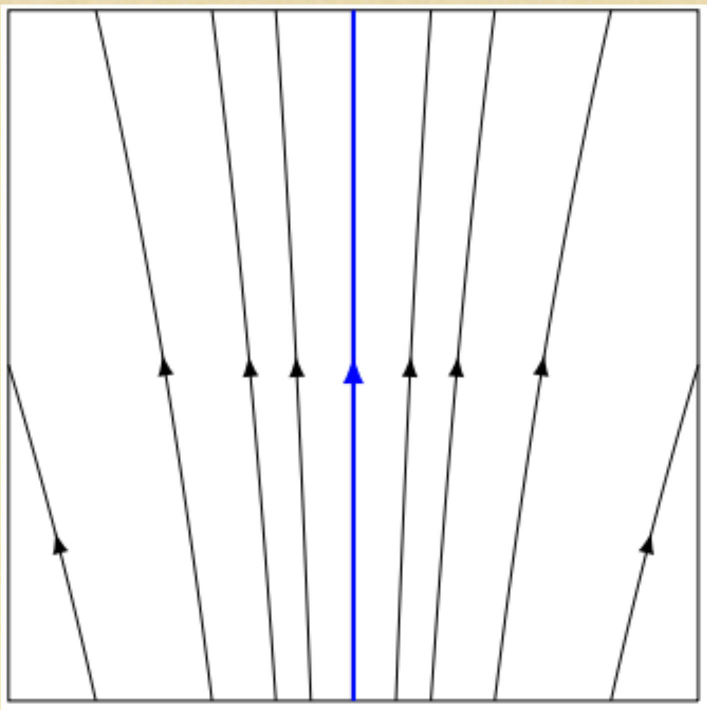


$S^1 \times I$

§ 3 Explicit manipulations

Prop. A Liouville domain (W, λ) is Weinstein iff

(1) Every flowline of X_λ converges to a C.P. in backward time.



$S^1 \times I$

(2) The flowlines of X_λ include no loops, broken or otherwise.

(3) Around each of its C.P.s, X_λ is gradient-like for some Morse function.

§ 3 Explicit manipulations

Given a Liouville domain (W, λ) and a sm. function $F : W \to \mathbb{R}$ with $F|_{\partial W} \equiv 0$,

$$\lambda_t = \lambda + t\,dF, \quad t \in [0,1],$$

is a L'ville h'topy on W. "Graphical perturbation"

Strategy: Cleverly choose F to control X_1.

§3 Explicit manipulations

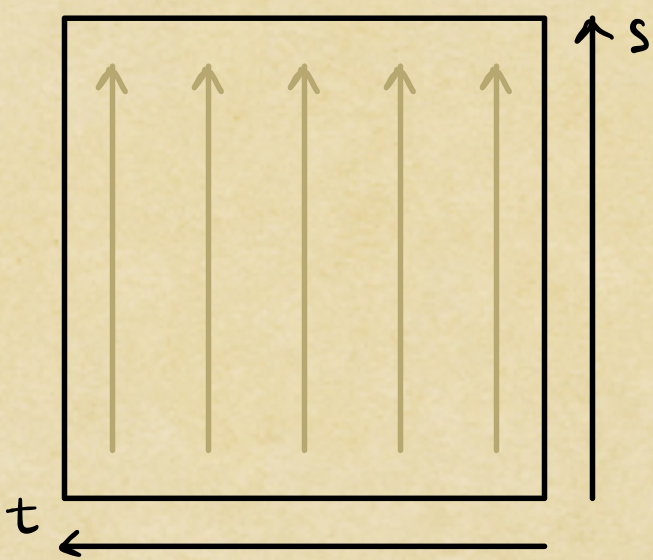
Given a Liouville domain (W, λ) and a sm. function $F: W \to \mathbb{R}$ with $F|_{\partial W} \equiv 0$,

$$\lambda_t = \lambda + t\,dF, \quad t \in [0,1],$$

is a L'ville h'topy on W. "Graphical perturbation"

Strategy: Cleverly choose F to control X_1.

Ex The 2D domain $[0, s_0] \times [0, t_0]$ with $\lambda = e^s dt$ has



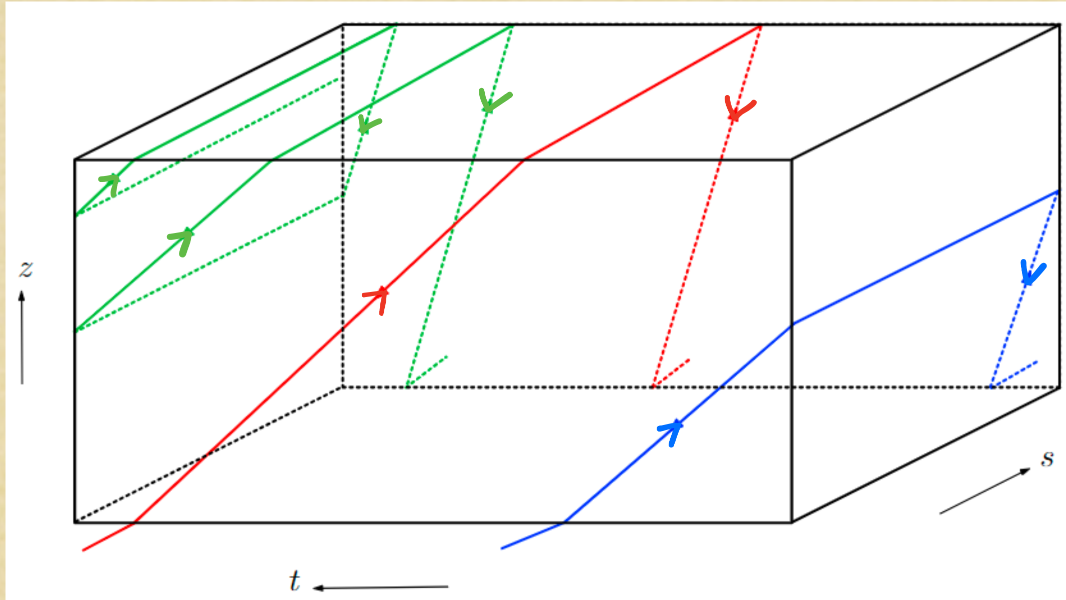
$X_0 = \partial_s$ and, given F,

$$e^s X_1 = (e^s + F_t)\,\partial_s - F_s\,\partial_t.$$

We'll carefully control these.

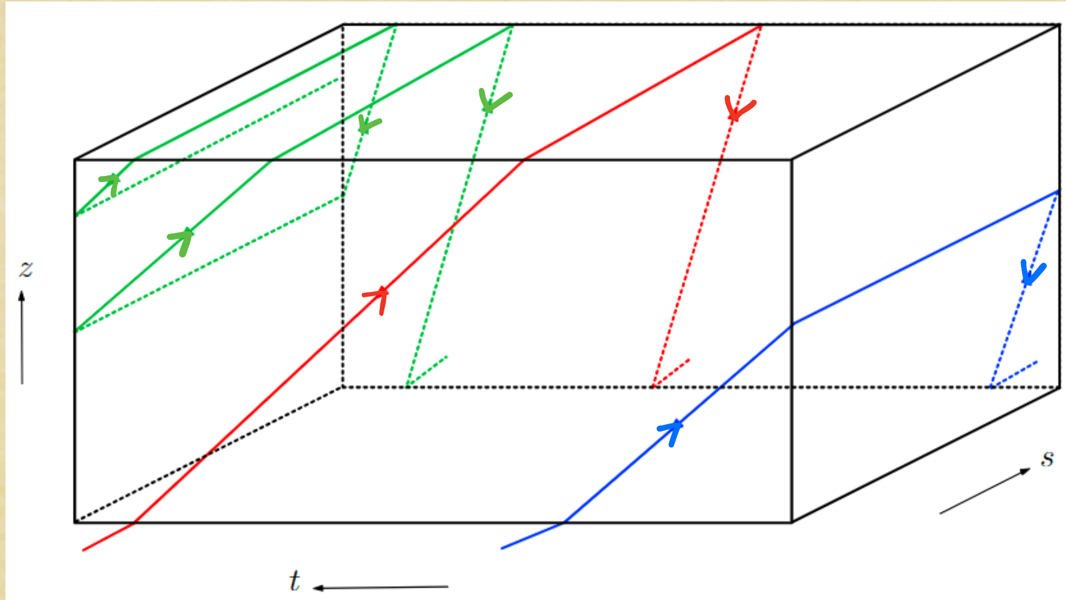
§3 Explicit manipulations

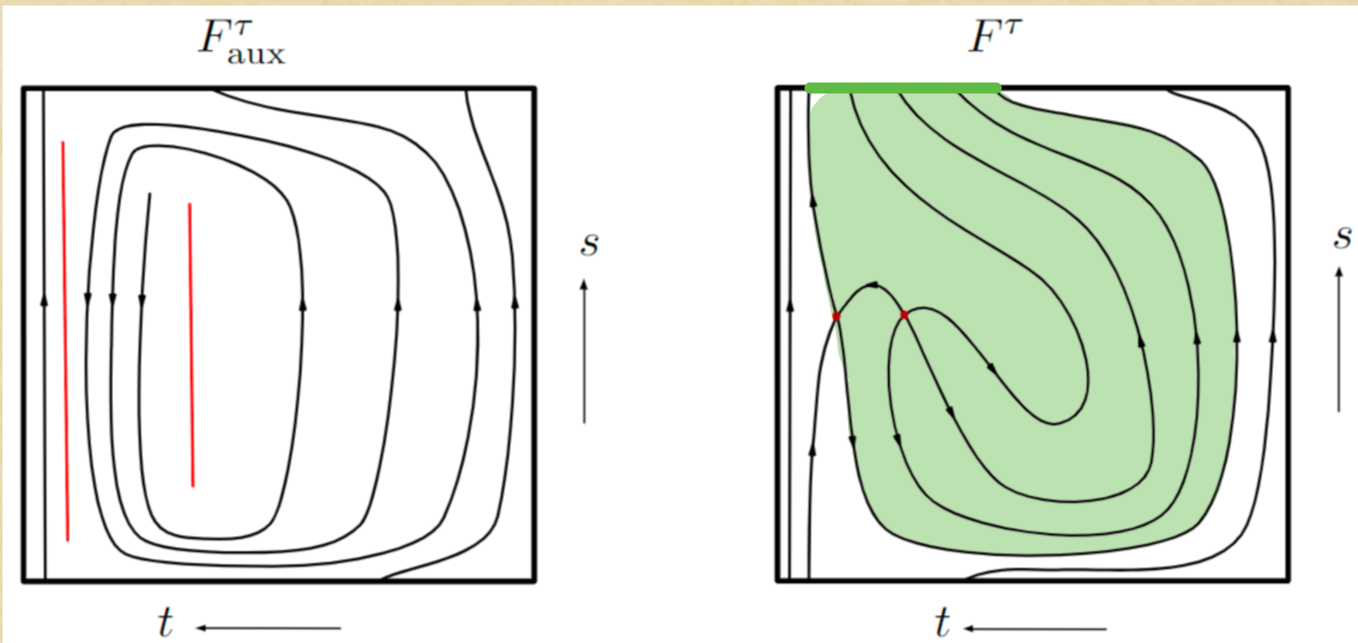
Ex (The box fold)



§3 Explicit manipulations

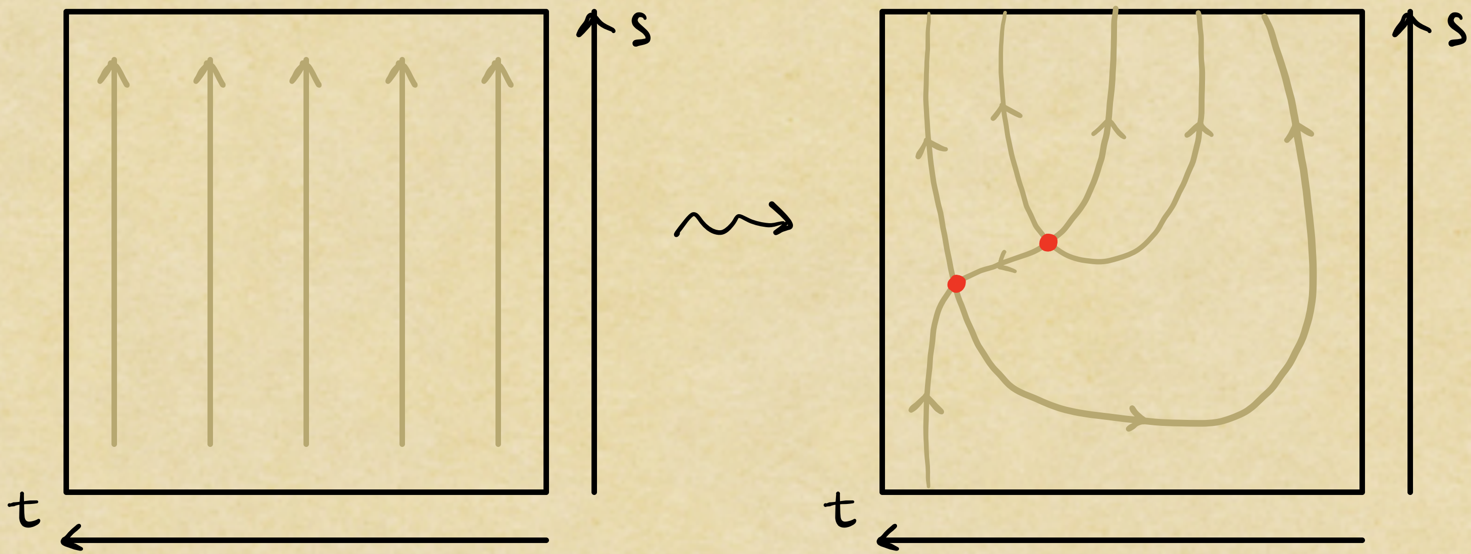
Ex (The box fold)

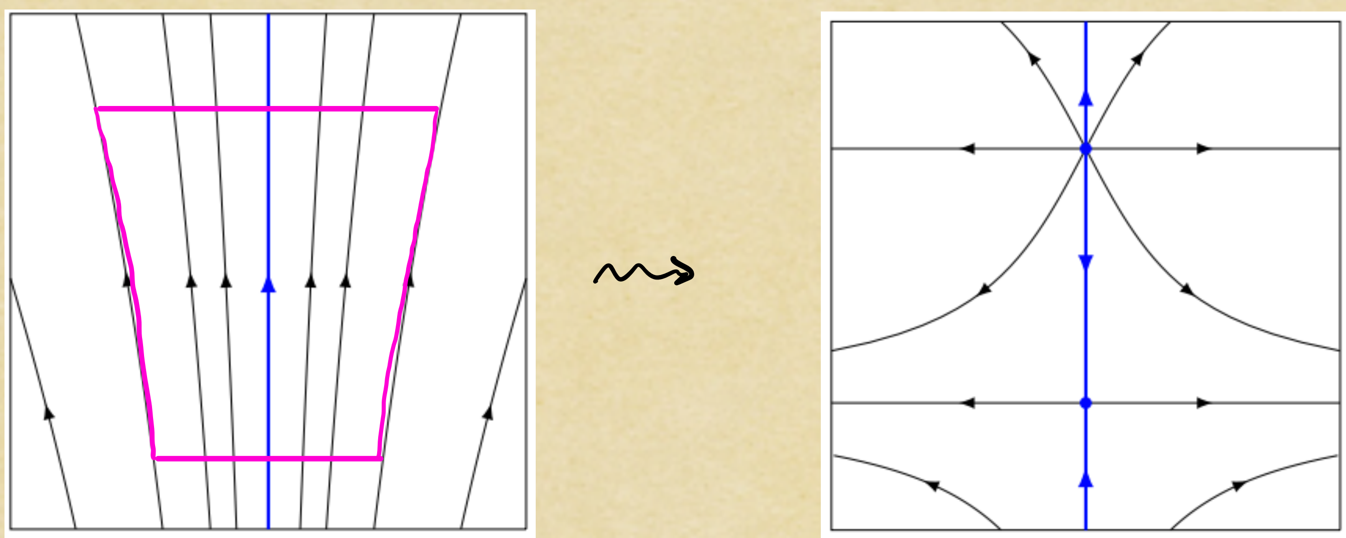




§3 Explicit manipulations

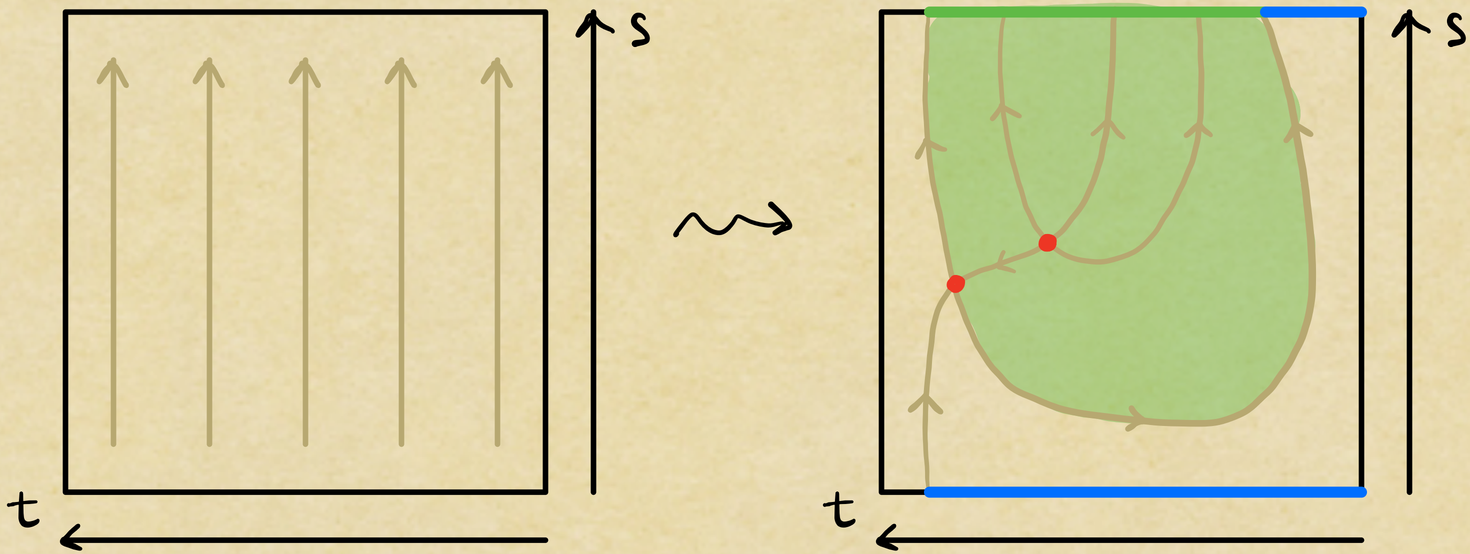
Ex. Cartoonish version:





$S^1 \times I$ $S^1 \times I$

§ 3 Explicit manipulations



In most settings, the box fold will not suffice — notice the drastic holonomy which affects untrapped points.

In general, there is a tension between

trapping and holonomy.

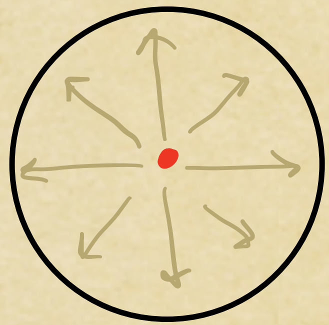
§4 Stably Weinstein Liouville domains

In general, there is a tension between

trapping and holonomy.

Stabilizing can give us room to balance these competing needs.

Recall: The 1-stabilization of (W^{2n}, λ) is $(W \times D^2, \lambda + \lambda_{std})$.



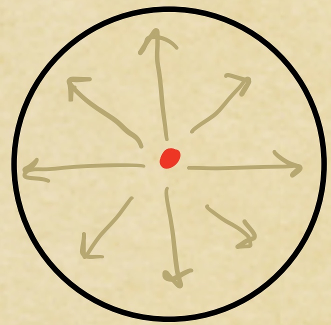
§4 Stably Weinstein Liouville domains

In general, there is a tension between

trapping and holonomy.

Stabilizing can give us room to balance these competing needs.

<u>Recall</u>: The 1-stabilization of (W^{2n}, λ) is $(W \times D^2, \lambda + \lambda_{std})$.



<u>Eliashberg–Ogawa–Yoshiyasu '21</u>: Every Liouville manifold is stably (flexible) Weinstein.

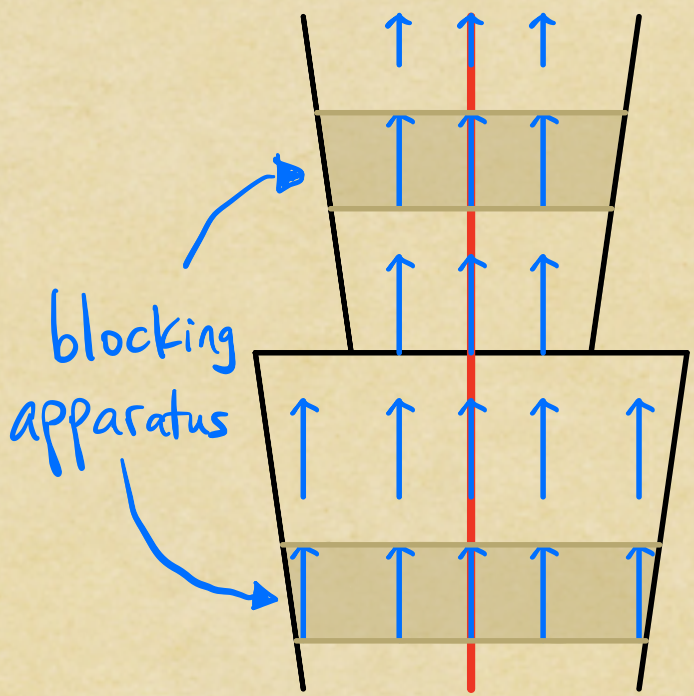
← The Liouville homotopies produced by EOY are not cpctly supported, so don't apply to Liouville domains.

§4 Stably Weinstein Liouville domains

Thm (Breen–C) Let $(W^{2n}_A, e^s\alpha)$ be a torus bundle Liouville domain, as constructed by Mitsumatsu, Geiges, and Huang. The 1-stabilization

$$(W_A \times D^2, e^s\alpha + \lambda_{std})$$

is Liouville homotopic to a Weinstein domain.



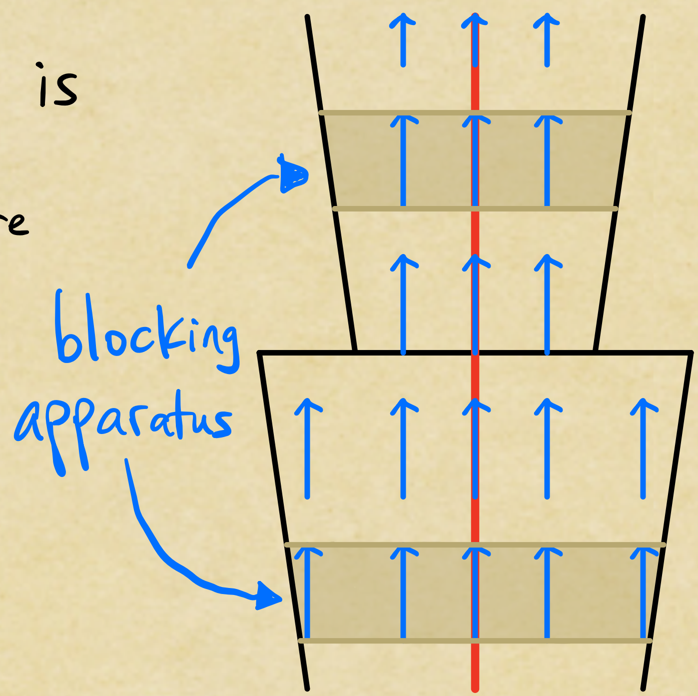
§4 Stably Weinstein Liouville domains

Thm (Breen–C) Let $(W_A^{2n}, e^s\alpha)$ be a torus bundle Liouville domain, as constructed by Mitsumatsu, Geiges, and Huang. The 1-stabilization

$$(W_A \times D^2, e^s\alpha + \lambda_{std})$$

is Liouville homotopic to a Weinstein domain.

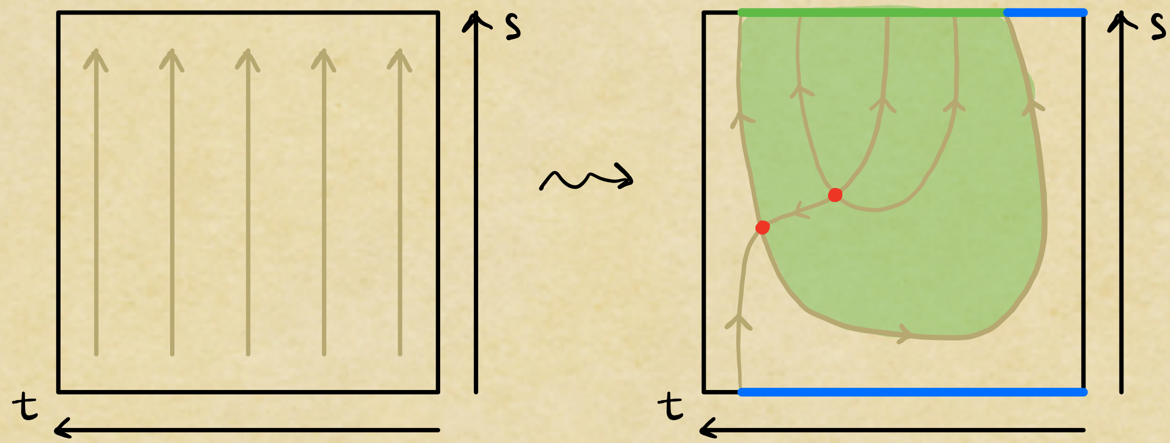
Remark. The domain $W_A \times D^2$ is diffeomorphic to DT^*M_A, where M_A is the mapping torus of $A: T^n \to T^n$, but these domains are not symplectomorphic.



§4 Stably Weinstein Liouville domains

Toolkit

① Chimney folds



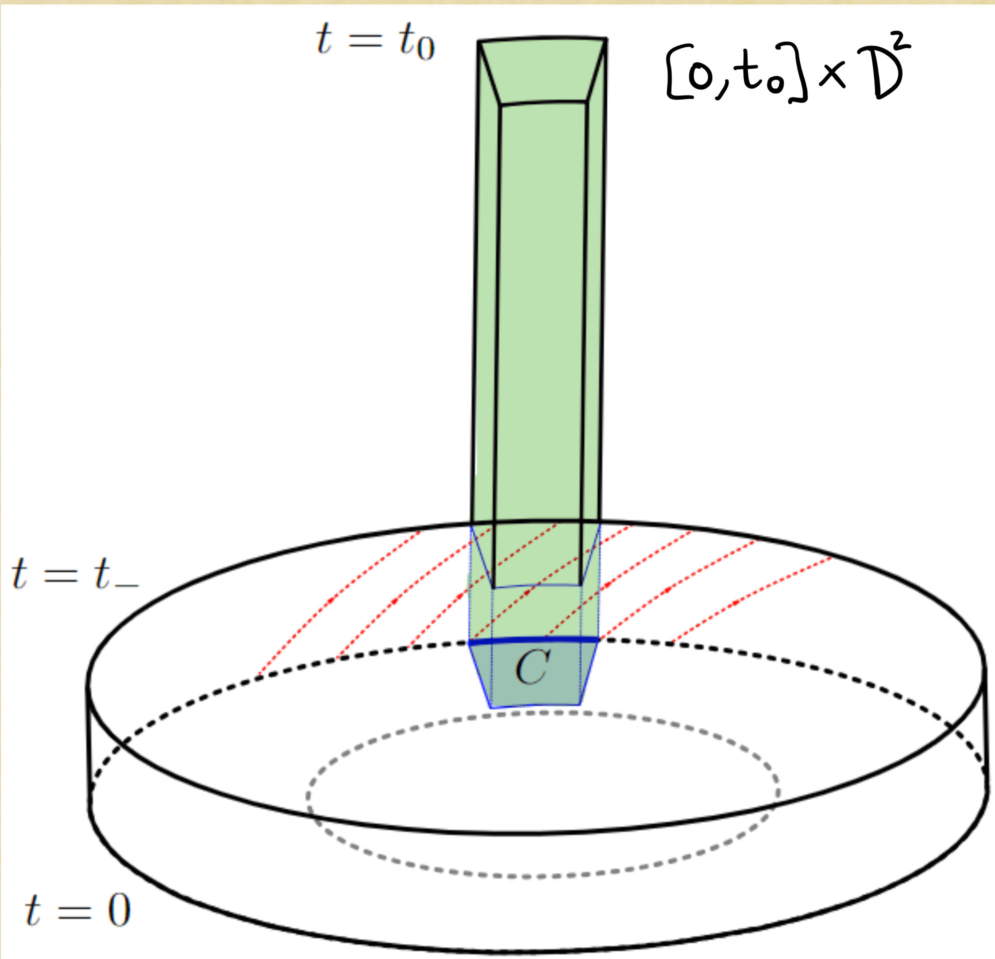
Box folds are built using the indicator function $F: W^{2n} \to \mathbb{R}$ of the symplectization of a contact handle body $H^{2n-1} \subset W$.

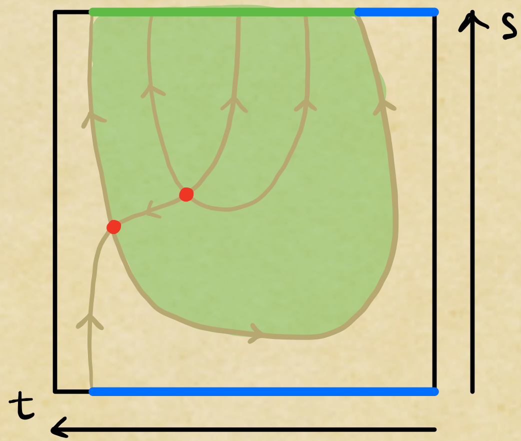
We'll now build a plug based on a more complicated contact-type hypersurface-with-bdry in W.

§4 Stably Weinstein Liouville domains

Toolkit

① Chimney folds





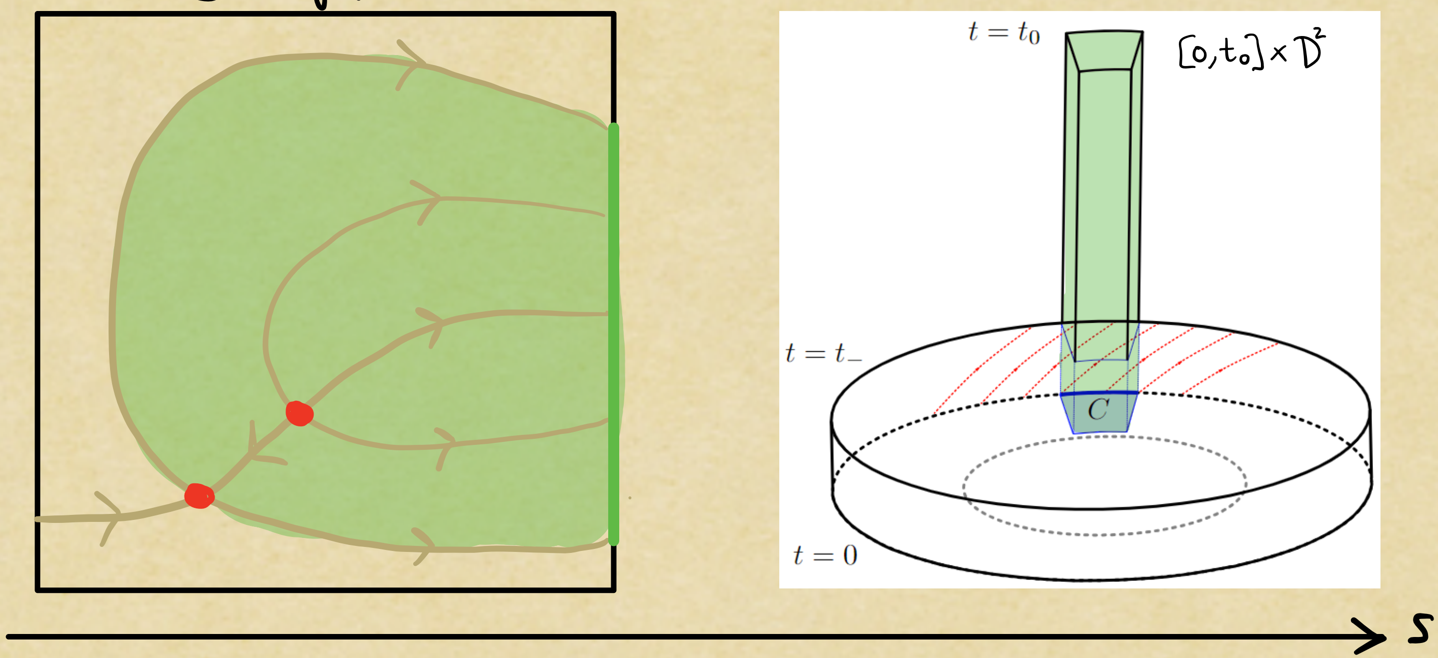
Most flowlines entering through the chimney portion are trapped in backward time.

Those which aren't exit thru the stove portion.

§4 Stably Weinstein Liouville domains

Toolkit

② Blocking apparatus(es)

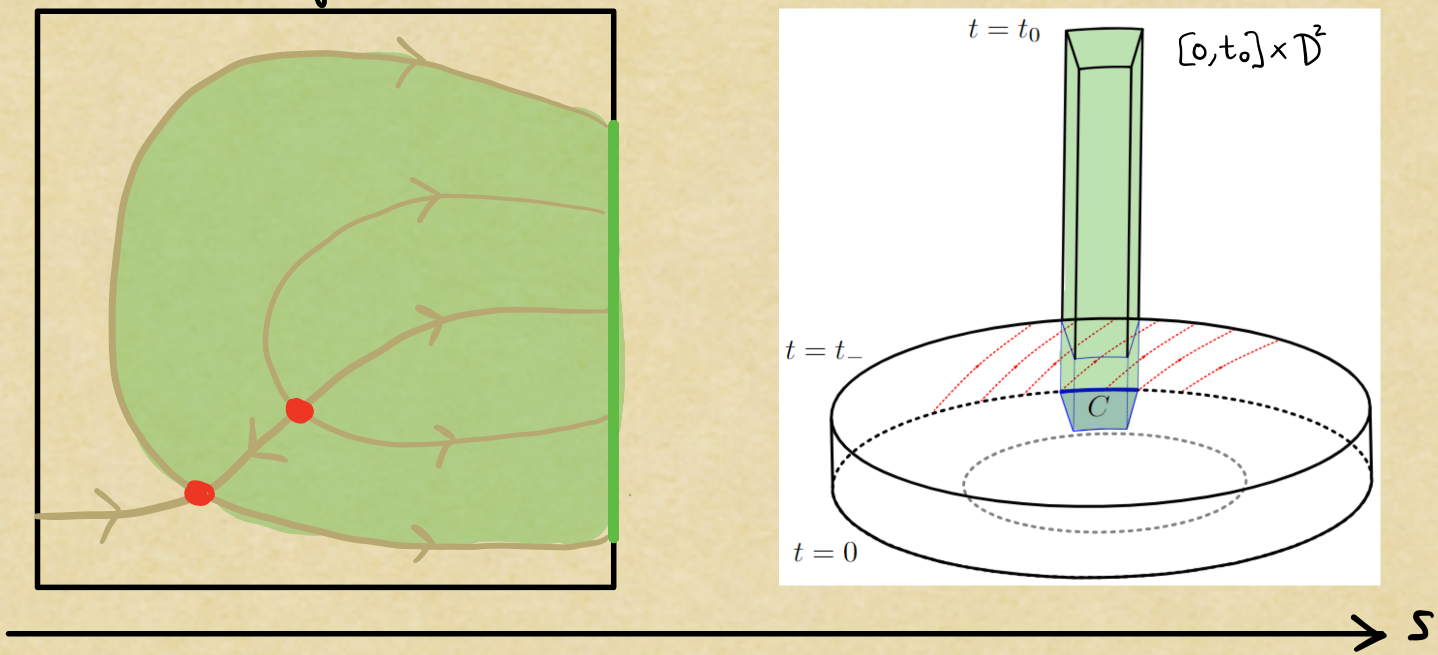


By installing a box hole behind a chimney fold, we trap those flowlines which entered in the chimney portion and exited in the stove region.

§4 Stably Weinstein Liouville domains

Toolkit

② Blocking apparatus(es)



Upshot: The stabilization direction gives an outlet for the holonomy which is forced by trapping.

To do

- Install blocking apparatuses on any stabilized Liouville domain to obtain a Weinstein str. ?
- Adapt the blocking apparatus to non-stabilized settings ?

Thanks!