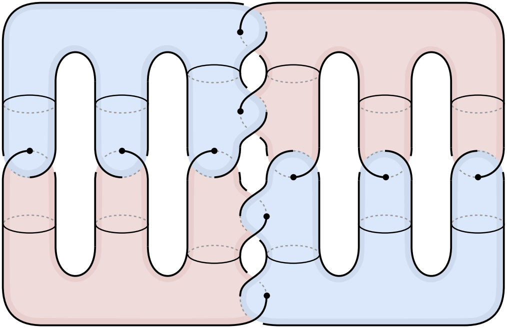
An atomic approach to Wall-type stabilization problems



Kyle Hayden

December 10, 2023

What makes smooth 4-manifolds special?

- Not governed by geometry or homotopy theory as in other dimensions (e.g., exotic $\mathbb{R}^4$'s)
- Can admit infinitely many smooth structures
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$X_0, X_1 =$ smooth, closed, simply-connected, orientable 4-manifolds

Theorem (Wall, 1964)

$$X_0 \underset{C^0}{\cong} X_1 \implies X_0 \#^k (S^2 \times S^2) \underset{C^\infty}{\cong} X_1 \#^k (S^2 \times S^2) \text{ for large } k \geq 0$$

Wall's Stabilization Problem

What's k? $\exists$ such X_i with $X_0 \# (S^2 \times S^2) \underset{C^\infty}{\not\cong} X_1 \# (S^2 \times S^2)$?

$k \approx$ some notion of distance between exotic 4-manifolds

Many analogs, including for

- all compact, orientable, exotic 4-manifolds (Gompf '84)
- exotic self-diffeomorphisms (Perron '86, Quinn '86)
- exotically knotted surfaces (Perron '86, Quinn '86 for $\# S^2 \times S^2$, Baykur-Sunukjian '15 for $\# T^2$)

How many stabilizations are required to dissolve exotica?

Burst of progress in recent years.

- "One is enough"-type (e.g., Auckly-Kim-Melvin-Ruberman-Schwartz '17)
- "One isn't enough"-type results (e.g., Lin '20, Lin-Mukherjee '21, Guth '22, Kang '22)

Today: Try to approach via failure of *h*-cobordism theorem.

1. **Construction:** Exotic phenomena that are candidates to survive stabilization, e.g., closed 4-manifolds with $\pi_1 = 1$.
2. **Proof of concept:** Exotic surfaces in B^4 that remain exotic after one internal stabilization.

Construction (H '20)

There exist exotic contractible Stein domains X_0, X_1 that are candidates to remain exotic after $\# S^2 \times S^2$.

X_i are branched double covers of B^4 along exotic disks $D_i \subset B^4$

$\Rightarrow X_i \# S^2 \times S^2$ are branched double covers of B^4 along $D_i \# T^2$

Theorem (H, '23)

The once-stabilized disks $D_i \# T^2$ remain exotically knotted in B^4, distinguished by Bar Natan homology $\widetilde{\mathrm{BN}}$ over $\mathbb{F}_2[H]$

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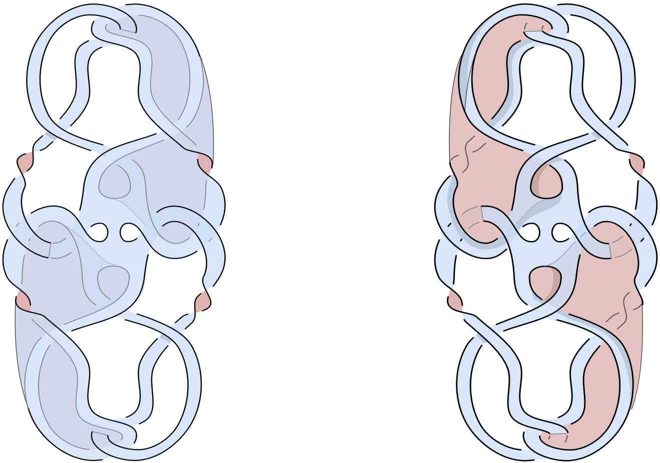

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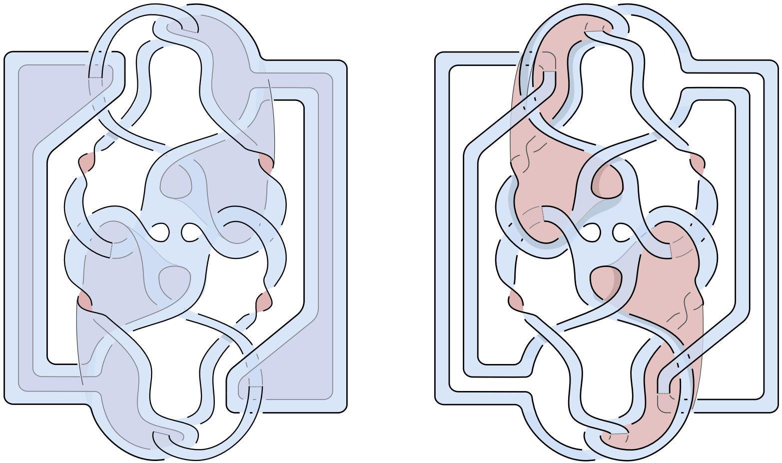

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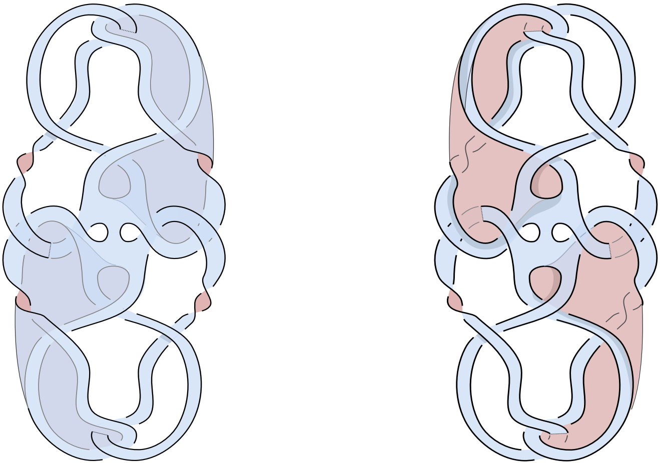
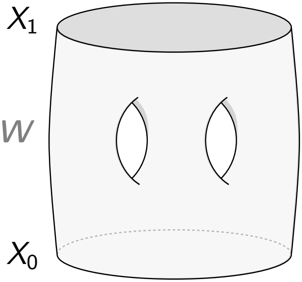


h-cobordisms and Wall-type stabilization problems

Let X_i be smooth, closed, orientable 4-manifolds with $\pi_1 = 1$.

Wall '64: X_i homotopy equivalent $\implies$ X_i are *h*-cobordant

5D cobordism $W : X_0 \to X_1$ with $X_i \hookrightarrow W$ homotopy equiv.

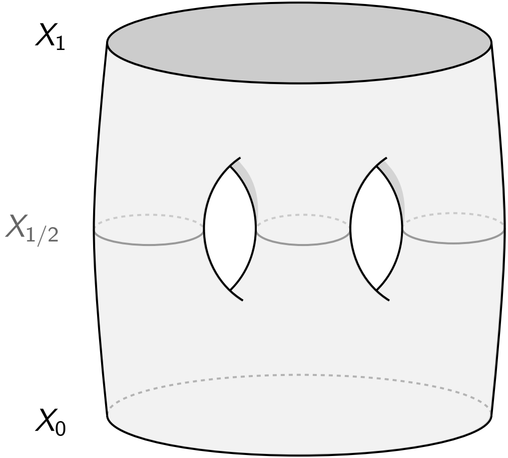


Q: Is $W \cong X_i \times [0,1]$?

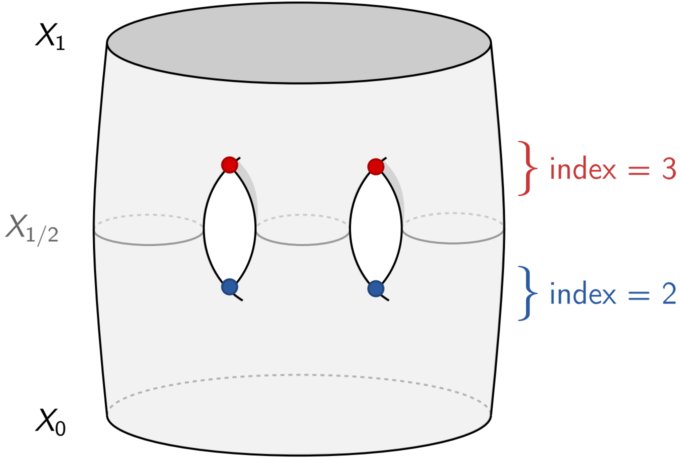
Freedman '82: Yes for C^0

Donaldson '87: No for C^∞

(exotic 4D phenomena ≈ failure of 4D h-cobordisms to simplify; higher dimensional exotica ≈ failure of h-cobordisms to exist)



Smale: Can eliminate all critical points except index $2, 3$.
These occur in algebraically canceling pairs. (h-cob!)



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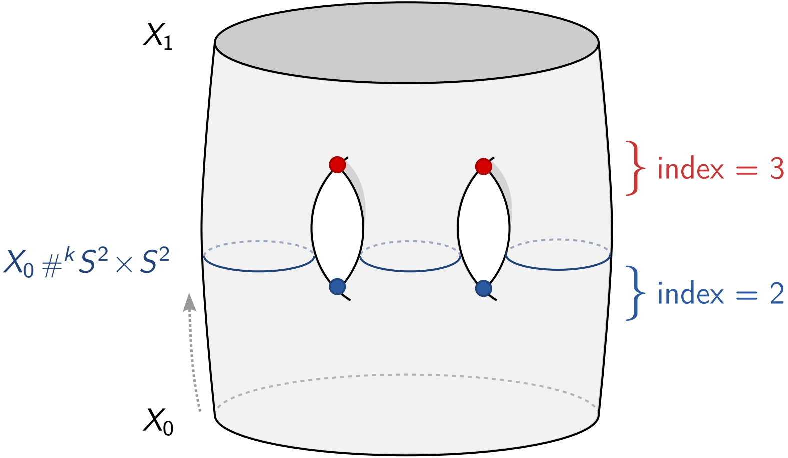

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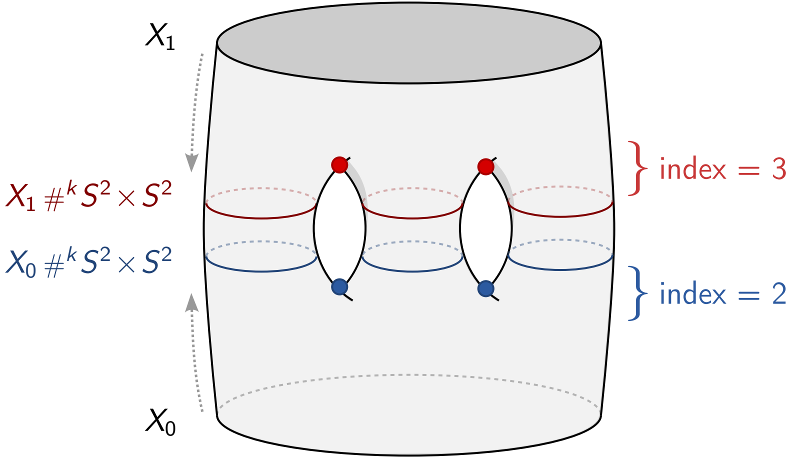

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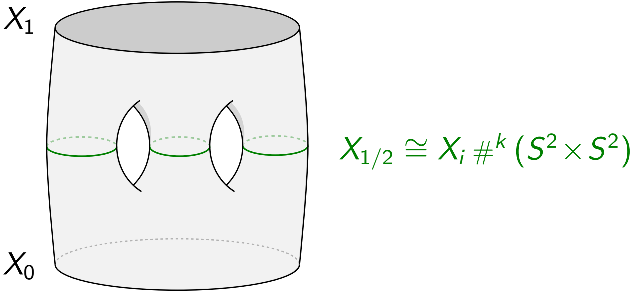
$$\Longrightarrow \; X_0 \#^k (S^2 \times S^2) \underset{C^\infty}{\cong} X_1 \#^k (S^2 \times S^2) \text{ for some } k \geq 0$$



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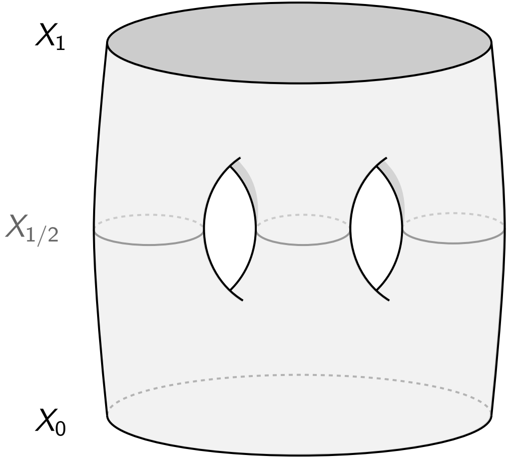
$k \approx$ some notion of distance between exotic 4-manifolds

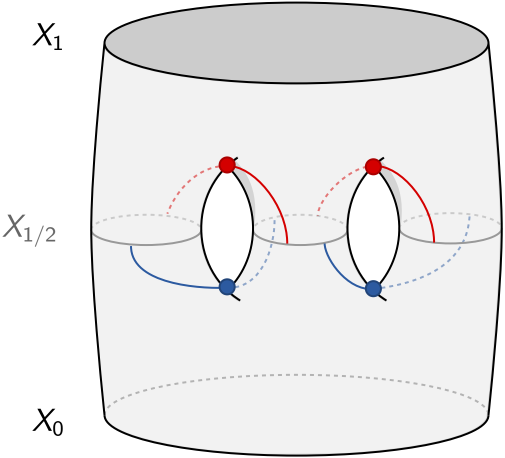
$\approx$ some notion of complexity for h-cobordisms

Anatomy of an h-cobordism

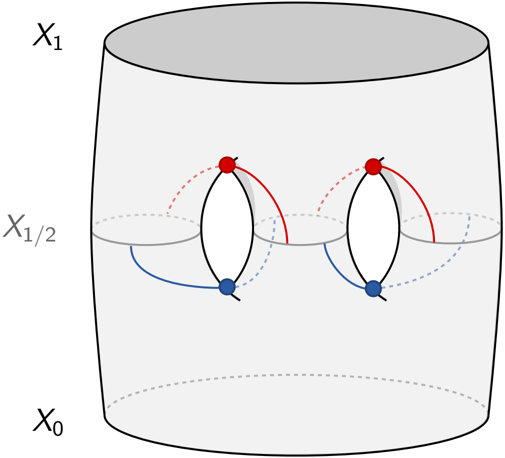


X_1
$X_{1/2}$
X_0

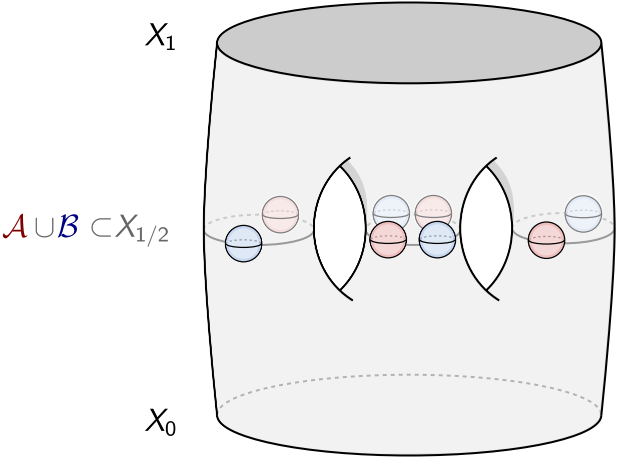
Critical points $\leftrightsquigarrow$



Critical points ⇜⇝ handles ⇜⇝



Critical points $\rightsquigarrow$ handles $\rightsquigarrow$ spheres in $X_{1/2}$

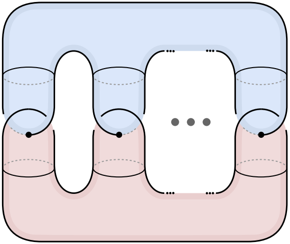


$\mathcal{A}$ = attaching spheres for 3-handles, $\mathcal{B}$ = belt spheres for 2-handles

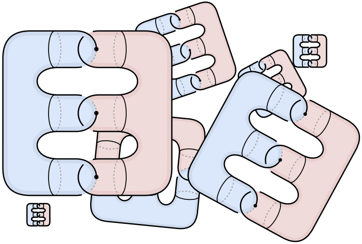
"Atomic" approach: Start with appropriate configuration of spheres, then build out into an *h*-cobordism.

What's the right type of complexity in our *h*-cobordism, in terms of these spheres $\mathcal{A} \cup \mathcal{B} \subset X_{1/2}$?

...spheres with many intersections?

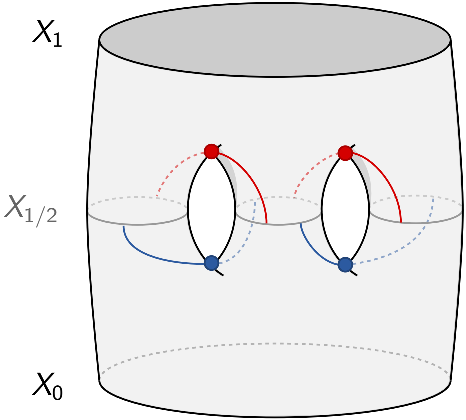


...many intersecting spheres?



The Recycling Problem

If the various ascending/descending manifolds have little interaction, then often need fewer stabilizations than critical points.



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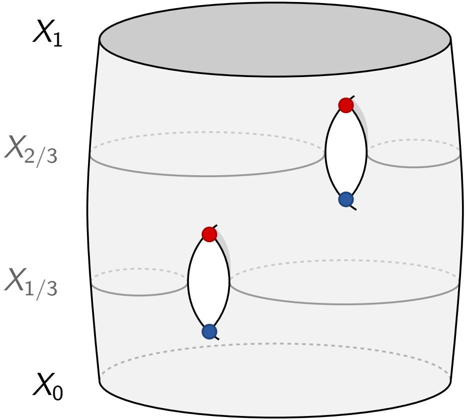

$X_{2/3} \cong X_1 \# S^2 \times S^2$

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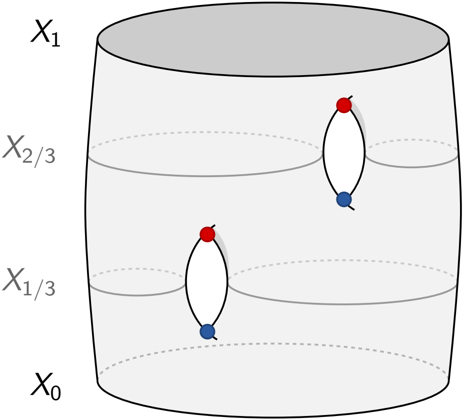

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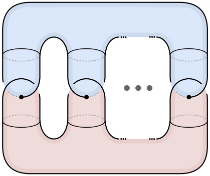
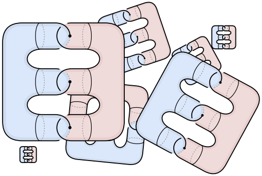
EX: If critical points can be reordered, often have $X_{1/3} \underset{C^\infty}{\cong} X_{2/3}$

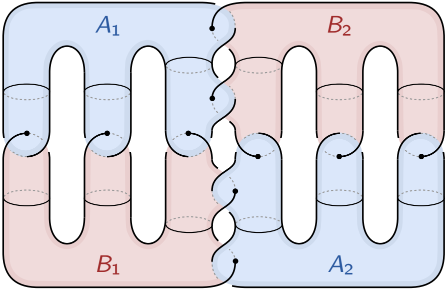
How to prevent recycling?

Any opinions, findings, or recommendations expressed on this slide do not necessarily reflect the views of the National Science Foundation.

 ?

 ?

Want $\mathcal{A}$ and $\mathcal{B}$ to intersect "completely":



Intersections $\mathcal{A} \cap \mathcal{B}$ prevent reordering of critical points.

Start with simple candidate 2-complex $\mathcal{A} \cup \mathcal{B}$.

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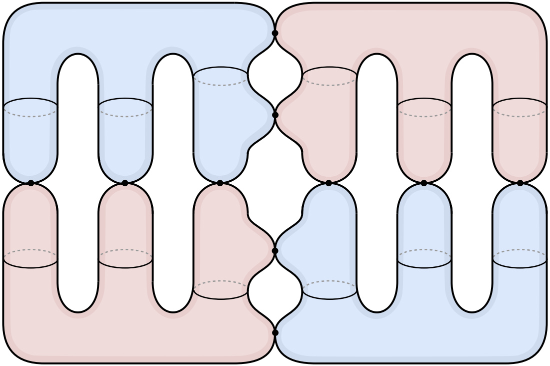
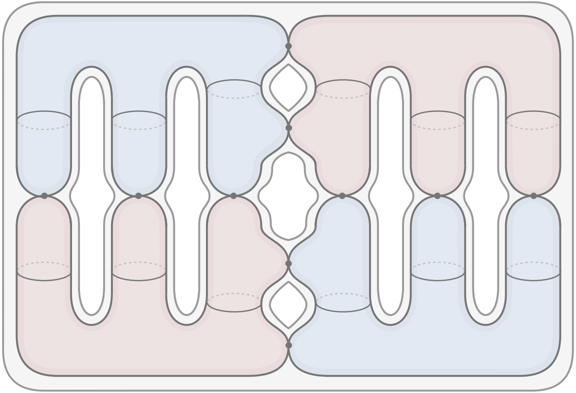


Start with simple candidate 2-complex $\mathcal{A} \cup \mathcal{B}$. Thicken to a 4D neighborhood (i.e., plumb copies of $S^2 \times D^2$).



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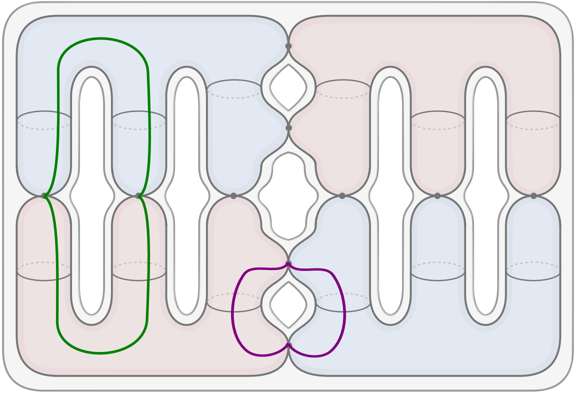

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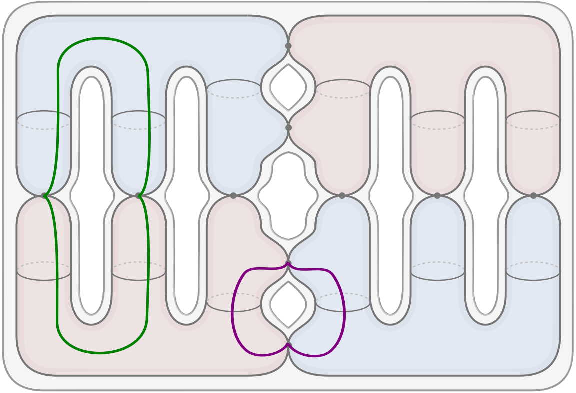

Attach 2-handles to kill π_1 (without killing 4D invariants).

Start with neighborhood of $\mathcal{A} \cup \mathcal{B}$

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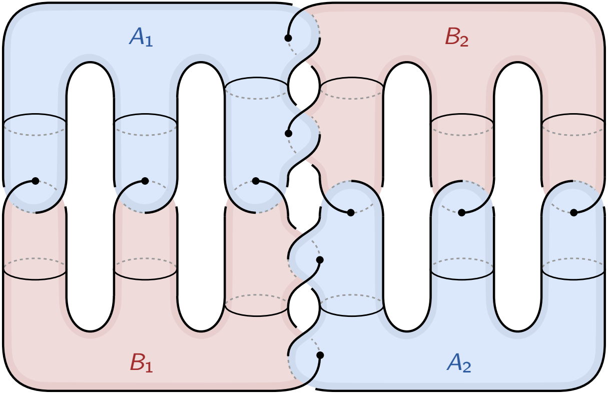

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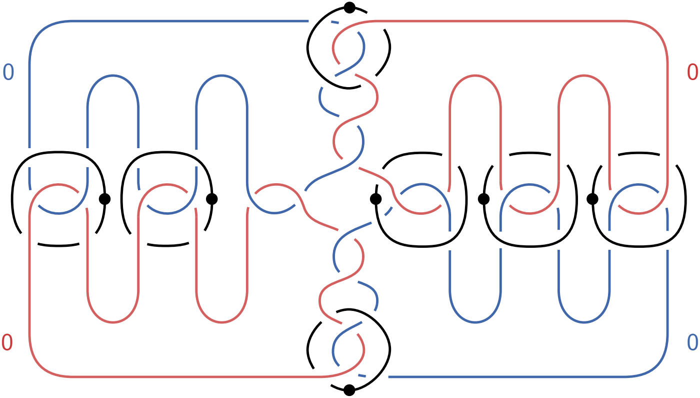
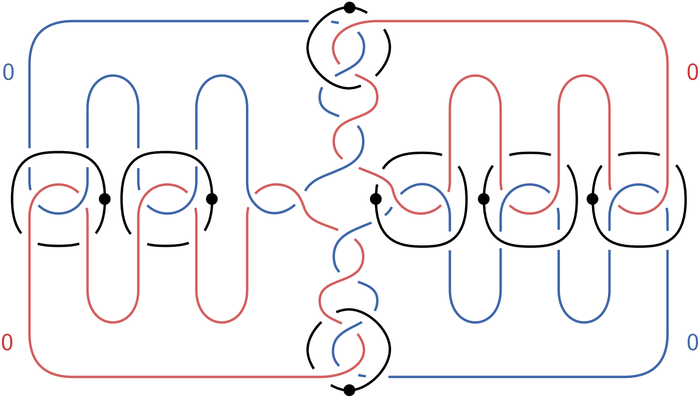


Start with neighborhood of $\mathcal{A} \cup \mathcal{B}$, then build out $X_{1/2}$.



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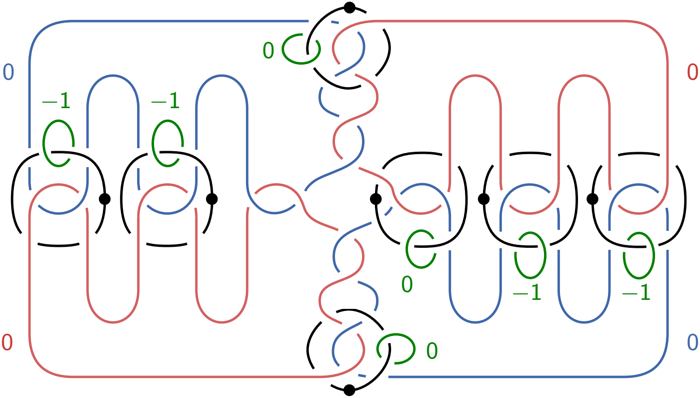

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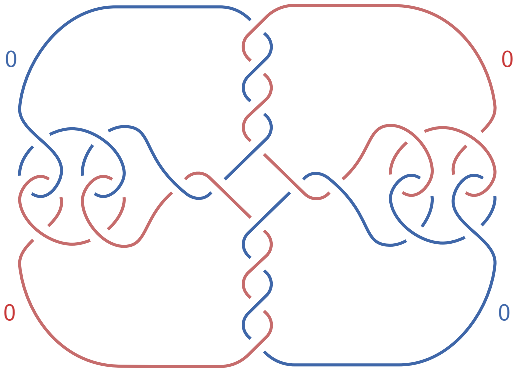

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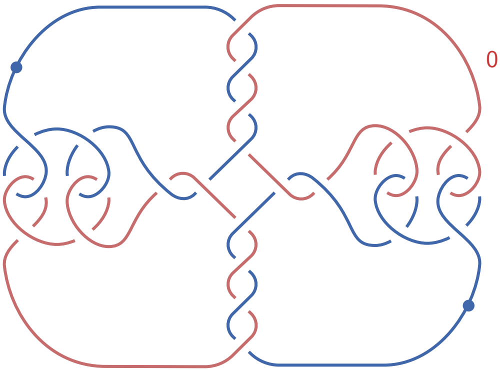

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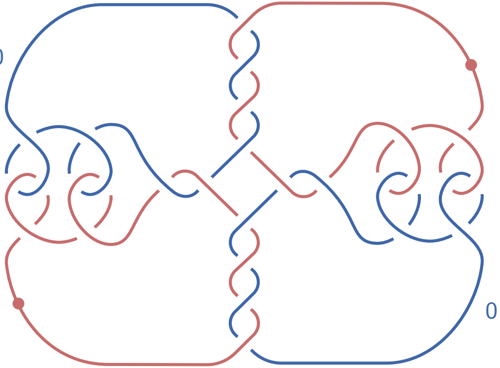

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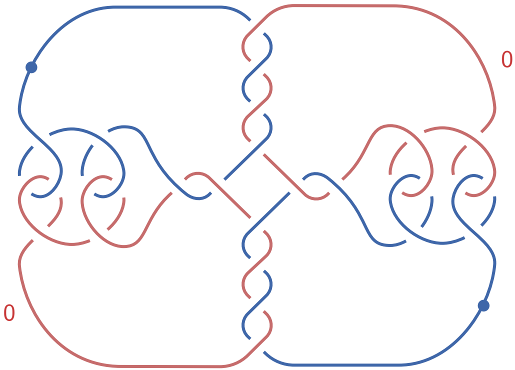

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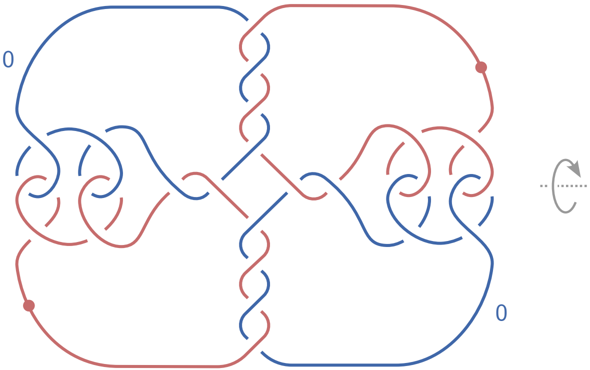
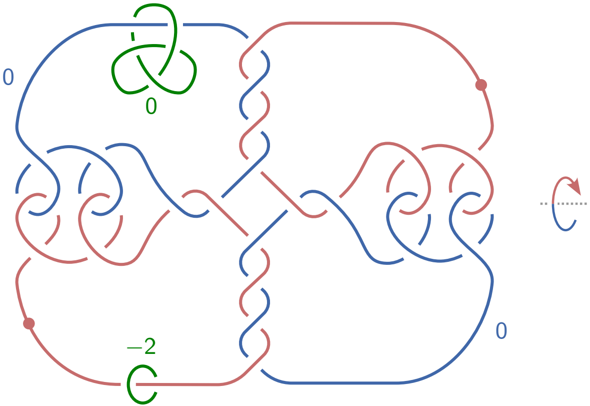
$X_1 =$ 

Theorem (H, '23)

There are exotic closed 4-manifolds built out of X_0 and X_1.

$Y_1 =$ 

Theorem (H, '23)

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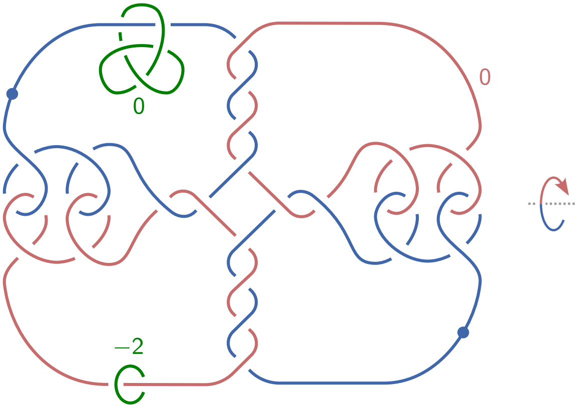

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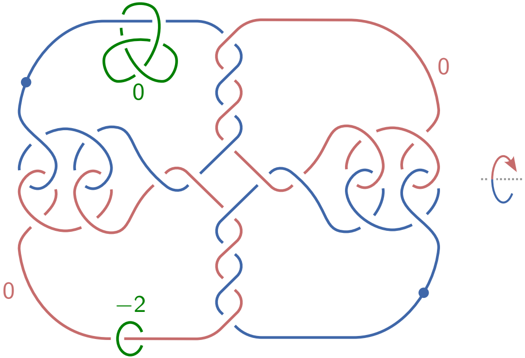
$Y_0 =$ 

Theorem (H, '23)

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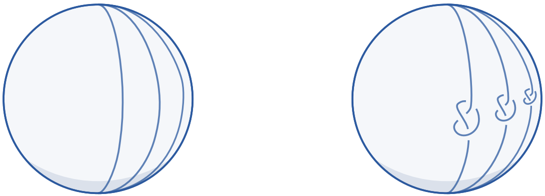
Do they remain exotic after $\# S^2 \times S^2$? And their "closures"?

Evidence from downstairs:

Surfaces in B^4 and stabilization

Smooth surfaces $S_0, S_1 \subset X$ are **exotically knotted** if

$$(X, S_0) \underset{C^0}{\cong} (X, S_1) \quad \text{but} \quad (X, S_0) \underset{C^\infty}{\ncong} (X, S_1).$$



(Not an actual example.)

Baykur-Sunukjian '15: Exotic surfaces are C^∞-equivalent after sufficiently many internal stabilizations $S \rightsquigarrow S \# T^2$.

Note: $\Sigma_2(X, S\#T^2) \cong \Sigma_2(X, S)\#(S^2 \times S^2)$

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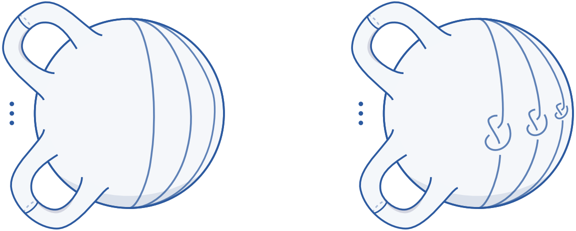

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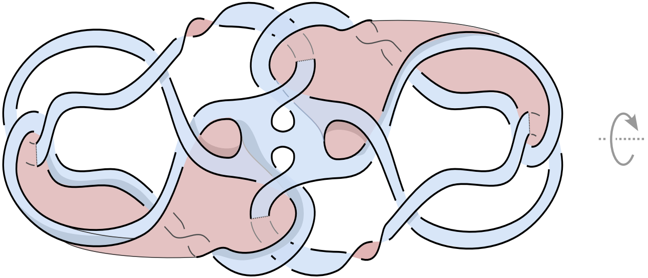
$D_1 =$ 

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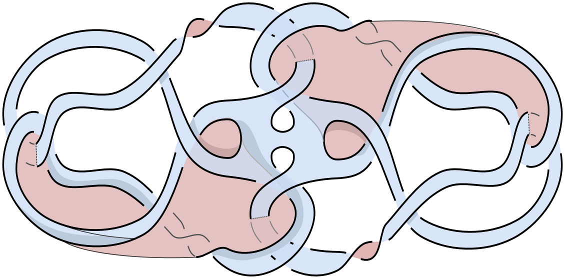
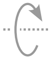
$D_1 =$ 



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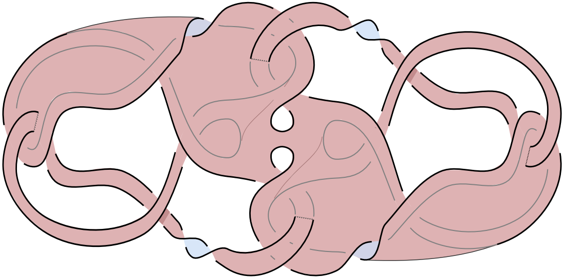

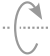

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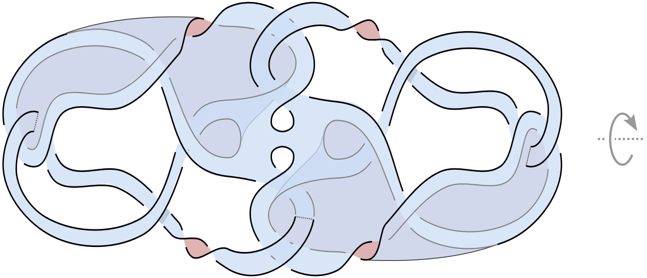
$D_0 =$ 

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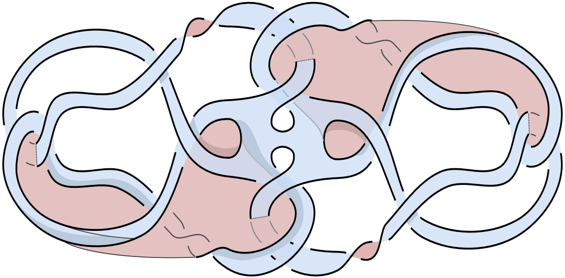
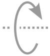
$D_1 =$  

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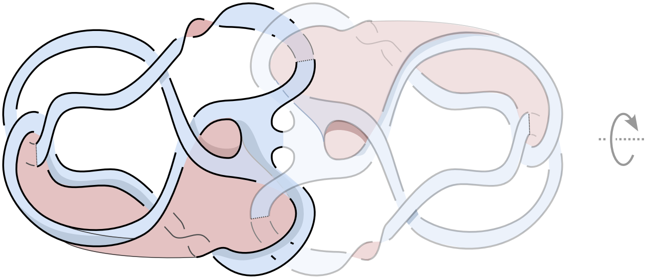

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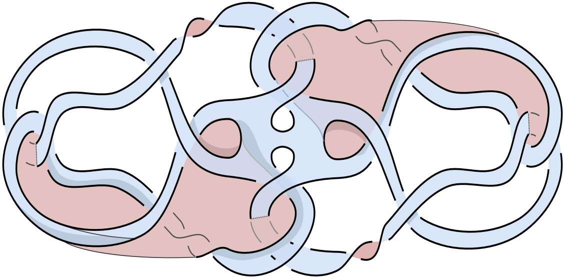
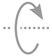
$D_1 =$ 



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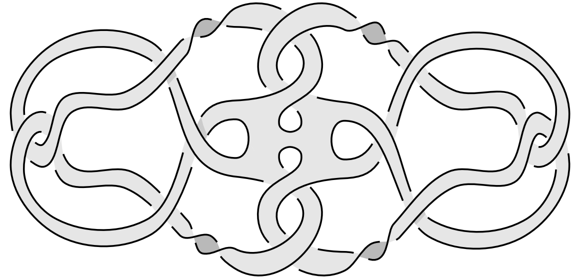
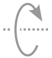
$D_{1/2} =$ 



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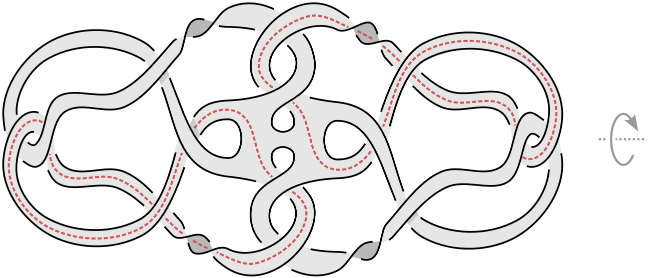
$D_{1/2} =$ 

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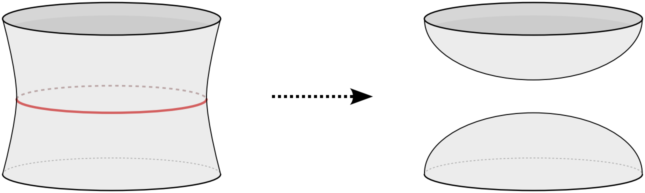

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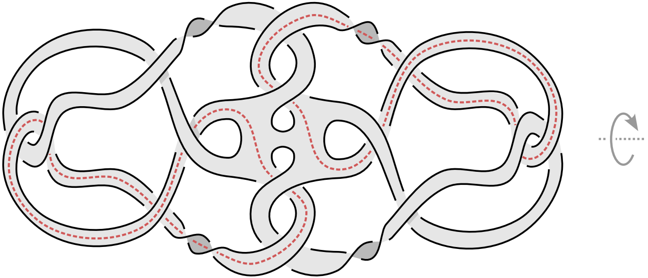
$D_{1/2} =$ 

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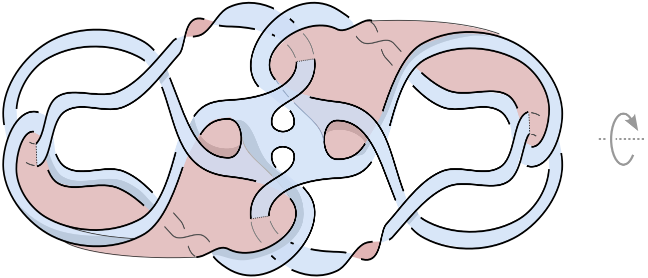
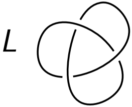
$D_1 =$ 

$X_i \# S^2 \times S^2$ are branched covers of B^4 along $D_i \# T^2$.

Theorem (H, '23)

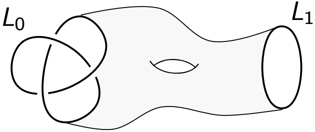
D_0 and D_1 remain distinct after one (internal) stabilization, inducing different maps on Bar-Natan homology over $\mathbb{F}_2[H]$.



$\rightsquigarrow \quad \mathrm{Kh}(L) = \bigoplus \mathrm{Kh}^{h,q}(L)$

(bigraded $\mathbb{F}_2$-vector space)

$\Sigma \subset S^3 \times [0,1]$



$\rightsquigarrow \; \mathrm{Kh}(\Sigma) : \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$

stabilization: $\mathrm{Kh}(\Sigma \# T^2) = 0$

Theorem (H, '23)

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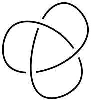
L  $\rightsquigarrow \widetilde{\mathrm{BN}}(L) = \bigoplus \widetilde{\mathrm{BN}}^{h,q}(L)$

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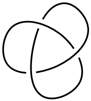
L  $\rightsquigarrow \widetilde{\mathrm{BN}}(L) = \bigoplus \widetilde{\mathrm{BN}}^{h,q}(L)$

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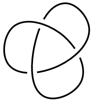
L  $\rightsquigarrow \widetilde{\mathrm{BN}}(L) = \bigoplus \widetilde{\mathrm{BN}}^{h,q}(L) \cong \left(\mathbb{F}_2[H]\right)^{|L|} \oplus \mathrm{Tors}$

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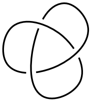
L  $\rightsquigarrow \widetilde{\mathrm{BN}}(L) = \bigoplus \widetilde{\mathrm{BN}}^{h,q}(L) \cong \big(\mathbb{F}_2[H]\big)^{|L|} \oplus \mathrm{Tors}$

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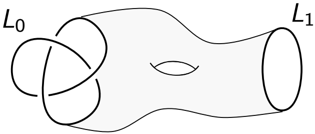

$\rightsquigarrow \widetilde{\mathrm{BN}}(\Sigma) : \widetilde{\mathrm{BN}}(L_0) \to \widetilde{\mathrm{BN}}(L_1)$

stabilization: $\widetilde{\mathrm{BN}}(\Sigma \# T^2) = H \cdot \widetilde{\mathrm{BN}}(\Sigma)$

Claim: $\widetilde{\mathrm{BN}}(D_0) \neq \widetilde{\mathrm{BN}}(D_1)$

- Puncture D_i and view as concordance $U \to K$
- Manually distinguish induced maps on Khovanov homology

$$\widetilde{\mathrm{Kh}}(D_i) : \mathbb{F}_2 \cong \widetilde{\mathrm{Kh}}(U) \to \widetilde{\mathrm{Kh}}(K)$$

(Uses approach developed with Sundberg in 2021)

- Lift to Bar Natan homology

$$\begin{array}{ccc} \widetilde{\mathrm{CBN}}(U) & \xrightarrow{\widetilde{\mathrm{CBN}}(D_0) - \widetilde{\mathrm{CBN}}(D_1)} & \widetilde{\mathrm{CBN}}(K) \\ \big\downarrow \pi & & \big\downarrow \pi \\ \widetilde{\mathrm{CKh}}(U) & \xrightarrow{\widetilde{\mathrm{CKh}}(D_0) - \widetilde{\mathrm{CKh}}(D_1)} & \widetilde{\mathrm{CKh}}(K) \end{array}$$

Claim: $\widetilde{\mathrm{BN}}(D_0) \neq \widetilde{\mathrm{BN}}(D_1)$

- Puncture D_i and view as concordance $U \to K$
- Manually distinguish induced maps on Khovanov homology

$$\widetilde{\mathrm{Kh}}(D_i) : \mathbb{F}_2 \cong \widetilde{\mathrm{Kh}}(U) \to \widetilde{\mathrm{Kh}}(K)$$

(Uses approach developed with Sundberg in 2021)

- Lift to Bar Natan homology

$$\begin{array}{ccc} \widetilde{\mathrm{BN}}(U) & \xrightarrow{\widetilde{\mathrm{BN}}(D_0)-\widetilde{\mathrm{BN}}(D_1)} & \widetilde{\mathrm{BN}}(K) \\ \big\downarrow \pi_* & & \big\downarrow \pi_* \\ \widetilde{\mathrm{Kh}}(U) & \xrightarrow{\widetilde{\mathrm{Kh}}(D_0)-\widetilde{\mathrm{Kh}}(D_1)} & \widetilde{\mathrm{Kh}}(K) \end{array}$$

Claim: $\widetilde{\mathrm{BN}}(D_0) \neq \widetilde{\mathrm{BN}}(D_1)$

- Puncture D_i and view as concordance $U \to K$
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$$\widetilde{\mathrm{Kh}}(D_i) : \mathbb{F}_2 \cong \widetilde{\mathrm{Kh}}(U) \to \widetilde{\mathrm{Kh}}(K)$$

(Uses approach developed with Sundberg in 2021)

- Lift to Bar Natan homology

$$\begin{array}{ccc} \mathbb{F}_2[H] = \widetilde{\mathrm{BN}}(U) & \xrightarrow{\widetilde{\mathrm{BN}}(D_0) - \widetilde{\mathrm{BN}}(D_1)} & \widetilde{\mathrm{BN}}(K) \\ \downarrow \pi_* & & \downarrow \pi_* \\ \mathbb{F}_2 = \widetilde{\mathrm{Kh}}(U) & \xrightarrow{\widetilde{\mathrm{Kh}}(D_0) - \widetilde{\mathrm{Kh}}(D_1)} & \widetilde{\mathrm{Kh}}(K) \end{array}$$

Claim: $\widetilde{\mathrm{BN}}(\Sigma_0 \# T^2) \neq \widetilde{\mathrm{BN}}(\Sigma_1 \# T^2)$

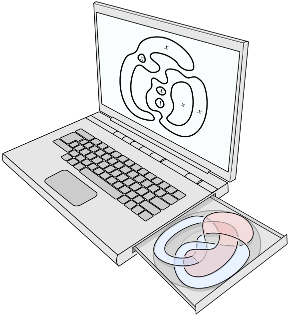
- Image of $\widetilde{\mathrm{BN}}(D_i)$ generated by image of $1 \in \widetilde{\mathrm{BN}}(U) \cong \mathbb{F}_2[H]$.

$$\Longrightarrow \; \delta := \widetilde{\mathrm{BN}}(D_0)(1) - \widetilde{\mathrm{BN}}(D_1)(1) \neq 0 \in \widetilde{\mathrm{BN}}(K)$$

- $\widetilde{\mathrm{BN}}(D_i \# T^2) = H \cdot \widetilde{\mathrm{BN}}(D_i) \implies$ Need to show $H \cdot \delta \neq 0$.

- δ lies in bigrading $\widetilde{\mathrm{BN}}(K)_{0,0}$

Computer calculation shows *every* nonzero element in $\widetilde{\mathrm{BN}}(K)_{0,0}$ survives multiplication by H. This includes δ. $\square$



The first two pages of the reduced Bar-Natan–Lee–Turner spectral sequence for the knot K, shown for $h \geq -4$ and $q \geq -12$:

Page 1

q \ h	$\dots$	-4	-3	-2	-1	0	1	2	3
2									2
0						2		3	
-2						2	6		
-4				2	13	14			
-6			4	24	19				
-8		13	44	24					
-10		75	28						
-12		26							
$\vdots$	$\iddots$								

Page 2

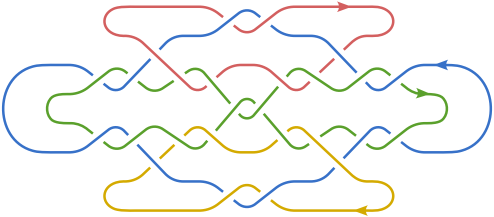
q \ h	$\dots$	-4	-3	-2	-1	0	1	2	3
2									
0						2			
-2									
-4					1				
-6									
-8									
-10									
-12									
$\vdots$	$\iddots$								

Connections to Floer homology

- F. Lin (2019): spectral sequence from (truncated) Bar-Natan homology to involutive monopole Floer homology
- Ladu (2022): monopole Floer homology of "protocork twists" (i.e., neighborhoods of configurations of 2-spheres)

Possible to prove twisting along X_0 and X_1 changes cobordism map on involutive Floer homology or related invariant?

Example: This L bounds $F, F' \subset B^4$ such that $X = \Sigma_2(B^4, F)$ and $X' = \Sigma_2(B^4, F')$ induce distinct maps on $\widehat{\mathrm{HF}}$.



Consider $\widehat{\mathrm{HFI}}$ over $\mathbb{F}_2[Q]/(Q^2)$. Using $\widetilde{\mathrm{BN}}(-L) \rightrightarrows \widehat{\mathrm{HFI}}(Y)$:

$$X \# S^2 \times S^2 \text{ and } X' \# S^2 \times S^2 \text{ induce distinct maps on } \widehat{\mathrm{HFI}}$$

$$\Longleftrightarrow$$

$$Y = \Sigma_2(L) \text{ satisfies } \dim_{\mathbb{F}_2}(Q \cdot \widehat{\mathrm{HFI}}(Y)) > 1$$

(bridge index = 4 $\Longrightarrow$ bfh_python *program might work!)*

Thank you!