Parabolic Representations and Signature

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joint with N. Dunfield

Ryley (70's):

Parabolic $\rho: \pi_1(\underbrace{S^3 \setminus K}_{E_K}) \to SL_2(\mathbb{R})$

m, ℓ commute

$\rho(m), \rho(\ell)$ commute

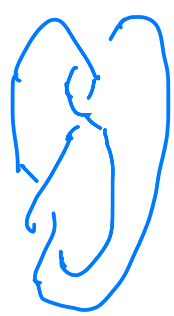
$\Rightarrow$ simultaniously "diagonalizable"

$$A\rho(m)A^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

$$A\rho(\ell)A^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

parabolic rep.s

$K = K_{p,q}$ rational



get Riley polynomial

$\rho_{p/q}(a)$

$\rho_{p/q}(0) \leftrightarrow \exists \rho$ with $\rho(m) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

Consequences: 1) E_{4_1} admits a complete hyperbolic str

since $\rho: \pi_1(E_{4_1}) \to SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$
$= \operatorname{Isom} \mathbb{H}^3$

2) Compute A-polynomial of 2-bridge knots

3) <u>Riley's Conjecture</u>: $\rho_{a/b}(q)$ has $\geq \frac{1}{2}|\sigma(K)|$

real roots

Gordon (2016) proved conjecture

each root gives

$\rho : \pi_1(E_K) \to SL_2(\mathbb{R})$

(*) signed count (with multiplicity) of real parabolic reps $= -\frac{1}{2}\sigma(K)$

<u>Th^{m} (Dunfield - R)</u> (*) holds if K is small alternating knots

small: no closed incompressible sfcs in E_K

(*) holds for $c(K) \leq 10$, small Montesinos knots

(*) <u>No</u> for $T(p,q)$ $\quad 1 > \frac{1}{p} + \frac{1}{q} + \frac{1}{2}$

<u>Character Varieties:</u> $X_G(Y) = \{\rho : \pi_1(Y) \to G\}/\sim$

$\rho_1 \sim \rho_2$ if $\operatorname{tr}\rho_1 = \operatorname{tr}\rho_2$

if $\rho_2(x) = A\rho_1(x)A^{-1} \Rightarrow \rho_1 \sim \rho_2$

so $\sim$ includes mod out by conjugation

<u>Ex:</u> $G = SU(2)$, $Y = T^2$

$\pi_1(T^2) = \mathbb{Z}^2 = \langle m, l \rangle$

$\rho(m), \rho(l)$ commute

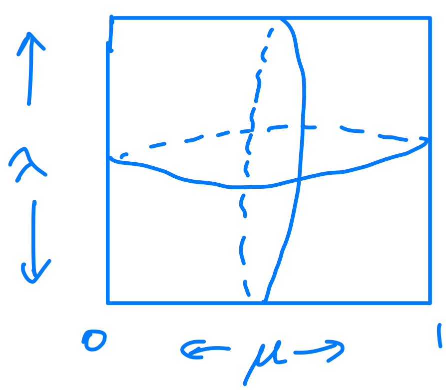
can be simultaneously diag.

$$\rho(m) = \begin{pmatrix} e^{i\pi\mu} & 0 \\ 0 & e^{-i\pi\mu} \end{pmatrix}$$

$$\rho(\lambda) = \begin{pmatrix} e^{i\pi\lambda} & 0 \\ 0 & e^{-i\pi\lambda} \end{pmatrix}$$

$$X_{SU(2)}(T^2) = S^1 \times S^1 \Big/ (\mu, \lambda) \sim (\mu^{-1}, \lambda^{-1})$$

$=$ "pillow case" orbifold

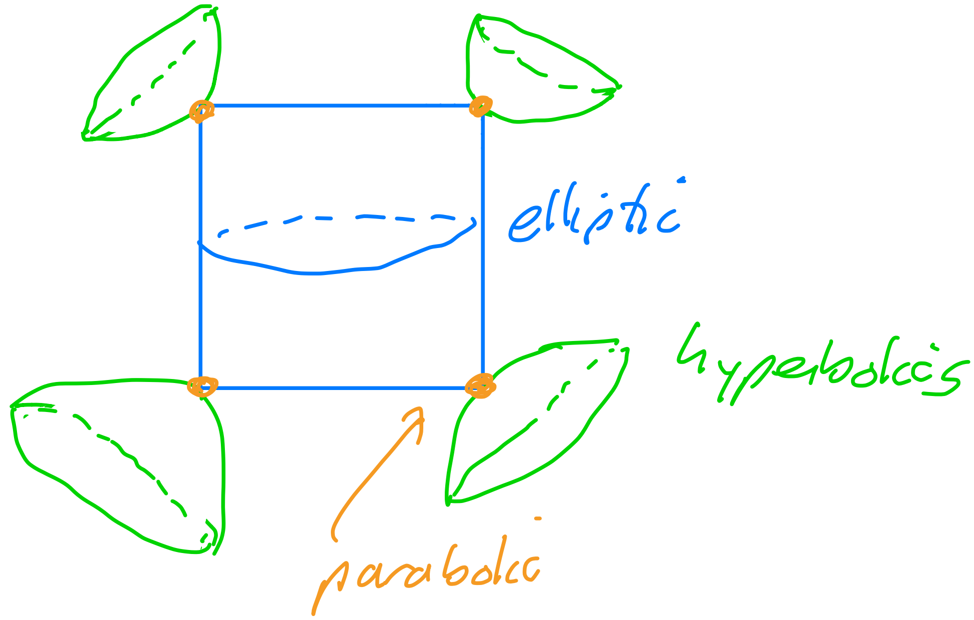


In $SL_2(\mathbb{R})$ 3 types of 1-param. subgroups

elliptic $\begin{pmatrix} \cos\pi\mu & -\sin\pi\mu \\ \sin\pi\mu & \cos\pi\mu \end{pmatrix}$

parabolic $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

hyperbolic $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$



$i: Y \to Z$

$i_*: \pi_1(Y) \to \pi_1(Z)$

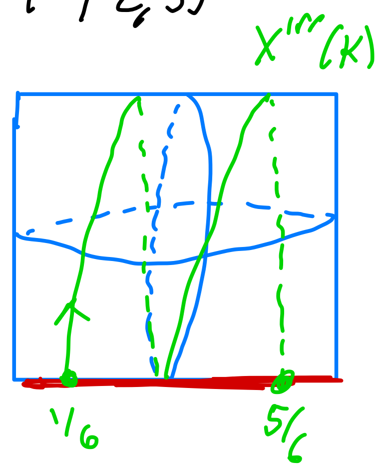
$i^*: X_G(Z) \to X_G(Y)$

$$i^*[\rho] = [\rho \circ i_*]$$

$i: \partial E_K \to E_K$

$i^*: X_G(E_K) \to X_G(\partial E_K) = X_G(T^2)$

<u>example:</u> $K = T(2,3)$



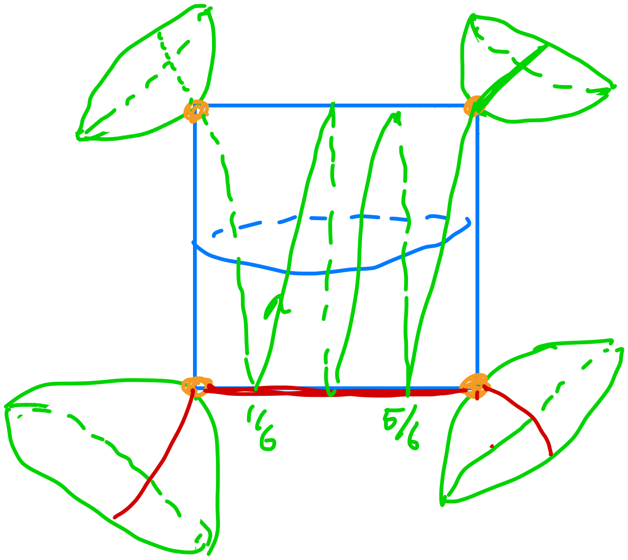
$i_*(X_{SU(2)}(K))$ is $X^{irr} \cup X^{red}$

reducible

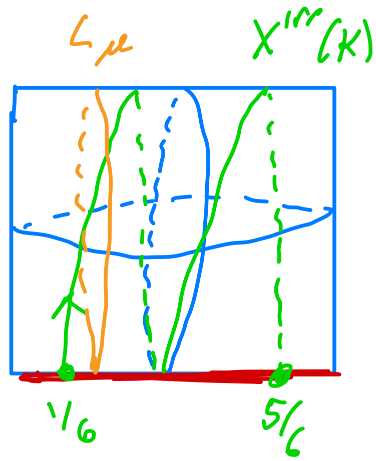
$X^{red}(K)$

same for any K

$$\pi_1(E_K) \to \underset{\mathbb{Z}}{\overset{}{\underset{\cong}{H_1(E_K)}}} \to G$$



Lin: $h^{K}_{SU(2)}(\mu) = L_\mu \cdot X^{irr}_{SU(2)}(K)$



Thm (Dunfield–R): if K is a small knot

$$h^{K}_{SU(2)}(\mu) + h^{K}_{E}(\mu) + h^{K}_{SL_2}(\mu) \equiv h(K) \quad \text{constant}$$

$SU(2) \xrightarrow{\text{projectivize}} \mathrm{Isom}(S^2)$, $E \xrightarrow{\text{proj}} \mathrm{Isom}(E^2)$, $SL_2 \xrightarrow{\text{projectivize}} \mathrm{Isom}(H^2)$

if something in $h^{K}_{SL_2}(\mu)$ "escape to ∞"
then get incompressible sfc.

Thm (Lin, Herald, Heusener–Kroll)

$$h^{K}_{SU(2)}(\mu) = -\tfrac{1}{2}\,\sigma_{e^{2\pi i \mu}}(K)$$

$$h^K_{SU(2)}(1/2) = -1/2\, \sigma(K)$$

observation: $\Delta_K(-1) \neq 0$ Alexander poly

$$h^K_{\mathbb{C}}(1/2) = 0$$

so $\circledast \Leftrightarrow h^K_{SL_2(\mathbb{R})}(1/2)$

if $X^{1/2, irr}_{SL_2(\mathbb{R})}(K) = \emptyset$, then true

L-space conjecture (Boyer - Gordon - Watson)

Y closed, oriented, prime 3-mfd $b_1(Y) = 0$

1) $\pi_1(Y)$ left orderable (LO) $\Leftrightarrow$ 2) Y not an L-space (NLS)

dim $\widehat{HF}(Y) > |H_1(Y)|$

$\rho \in X^{1/2, irr}_{SL_2(\mathbb{R})}(K) \rightsquigarrow \tilde{\rho} : \pi_1(\Sigma_2(K)) \to SL_2(\mathbb{R})$

$e(\rho) = 0$

$\downarrow$

$\hat{\rho} : \pi_1(\Sigma_2(K)) \to \widetilde{SL_2}$

$\downarrow$ Boyer - Rolfsen - Wiest

$\Sigma_2(K)$ is LO

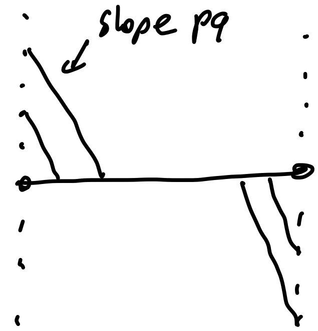
$\Sigma_2(K)$ is NLO if K is alternating

$\Rightarrow \circledast$ holds

More generally, expect ⊛ to hold whenever $\Sigma_2(K)$ is L-space

Culler–Dunfield: $T_{p,q}$

$X_{SL_2}(K)$ is



max height $\geq g(T(p,q)) - 1 = pq - q - p$

is $\frac{pq-q-p}{pq} > \frac{1}{2} \iff 1 > \frac{1}{2} + \frac{1}{p} - \frac{1}{q}$

<u>Prop</u>: $h(K) \equiv -\frac{1}{2}\sigma(K) \equiv \deg \operatorname{tr}_\mu : (X_{SL_2(\mathbb{R})}(K) \to \mathbb{P}^1)$

$(-1)^{h(K)} = \operatorname{sign} \Delta_K(-1)$