Math 4318 - Fall 2017
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 3, 5, 6, 7, 8, 14.* **Due: In class on September 21**

1. Compute $\int_1^3 x^2\,dx$ using the definition of integral. **Hint:** you can use either the Riemann or Darboux integral (since we know they are the same for bounded continuous functions on closed intervals). But one of them will probably be easier to compute.
2. Suppose $f : [a,b] \to \mathbb{R}$ is continuous and suppose that for every Riemann integrable function $g : [a,b] \to \mathbb{R}$ we have $\int_a^b f(x)g(x)\,dx = 0$. Show that $f(x) = 0$ for all $x \in [a,b]$.
3. Let $f : [0,1] \to \mathbb{R}$ be given by
$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ is in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
and $h : [0,1] \to \mathbb{R}$ be 1 for x rational and 0 for x irrational. Find a Riemann integrable function $g : [0,1] \to \mathbb{R}$ so that $g \circ f = h$. Notice that this shows that the composition of two integrable functions need not be integrable.
4. Prove the mean value theorem for integrals: If f is continuous on $[a,b]$ there there is a $c \in (a,b)$ such that
$$\int_a^b f(x)\,dx = f(c)(b-a).$$
5. Let f and g be continuous on $[a,b]$. If $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ then show there is some $c \in [a,b]$ such that $f(c) = g(c)$.
6. Compute the first derivatives of the following functions (and carefully justify your computations).
   (a) $F(x) = \int_0^{\sin x} \cos t^2\,dt$
   (b) $G(x) = \int_x^{x^2} \sqrt{1-t^2}\,dt$
   (c) $H(x) = \int_0^x xe^{t^2}\,dt$ (**Hint:** be careful on this one!)
7. If f is continuous on $[a,b]$ and $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$ for all $x \in [a,b]$ then show that $f(x) = 0$ for all $x \in [a,b]$.
8. Prove the integral version of the Taylor remainder: Suppose that f and its first $n+1$ derivatives are continuous on $[a,b]$ and $c \in (a,b)$. For each $x \in [a,b]$ we have that
$$f(x) = f(c) + f'(c)(x-c) + f''(c)/2(x-c)^2 + \ldots + \frac{f^n(c)}{n!}(x-c)^n + R_n$$
where
$$R_n = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt.$$
Hint: Use integration by parts and induction.
9. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $a > 0$ define
$$g(x) = \int_{x-c}^{x+c} f(x)\,dx.$$
Show that g is differentiable and compute $g'(x)$.

10. Show a subset of measure zero has measure zero.
11. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Set $C = \{x \in [a, b] : f'(x) = 0\}$ (such points are called **critical points**). Prove that $f(C)$ has measure zero. (This is the one dimensional version of **Sard's theorem**.) **Hint:** consider the function $F : [a, b] \times [a, b] \to \mathbb{R}$ defined for $x \neq y$ as $F(x, y) = \frac{f(y)-f(x)}{y-x} - f'(x)$ and for $x = y$ define $F(x, y) = 0$. Show F is continuous (and hence uniformly continuous). Given $\epsilon > 0$ use uniform continuity to show you can break $[a, b]$ into subintervals such that $|F(x, y)| < \epsilon$ for all x, y in the subintervals. Use this to show the image of intervals containing critical points have small length.
12. Recall that the **oscillation** of a function f at x is defined to be

$$\operatorname{osc}_x(f) = \lim_{\epsilon \to 0} \left(\sup\{f(y) : y \in [x - \epsilon, x + \epsilon]\} - \inf\{f(y) : y \in [x - \epsilon, x + \epsilon]\} \right).$$

Show that a function $f : (a, b) \to \mathbb{R}$ is continuous at $c \in (a, b)$ if and only if $\operatorname{osc}_c(f) = 0$.
13. Recall that if $C \subset \mathbb{R}$ then the characteristic function of C is χ_C which is 1 at all points in C and 0 elsewhere. Show that if $f : [a, b] \to \mathbb{R}$ is an integrable function and $c \in (a, b)$ then

$$\int_a^b f(x)\chi_{[a,c]}(x)\, dx = \int_a^c f(x)\, dx.$$

14. Prove or disprove the following statements:
    (a) $f \in \mathcal{R}([a, b]) \implies |f| \in \mathcal{R}([a, b])$
    (b) $|f| \in \mathcal{R}([a, b]) \implies f \in \mathcal{R}([a, b])$
    (c) $f \in \mathcal{R}([a, b]) \implies f^2 \in \mathcal{R}([a, b])$
    (d) $f^2 \in \mathcal{R}([a, b]) \implies f \in \mathcal{R}([a, b])$
    (e) $f^3 \in \mathcal{R}([a, b]) \implies f \in \mathcal{R}([a, b])$
    (f) $f^2 \in \mathcal{R}([a, b])$ and $f(x) \geq 0$ for all $x \in [a, b] \implies f \in \mathcal{R}([a, b])$
15. If $f : [a, b] \to \mathbb{R}$ is integrable show that

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx.$$