Math 4318 - Fall 2017
Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 4, 5, 7, 8, 9, 10.* **Due: In class on October 5**

1. Let
$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$
Show that $f'(0) = 1$ and that f is not strictly increasing in any neighborhood of 0. (Notice that this shows that a function have have positive derivative at a point without being increasing there. What does this say about the continuity of f' near 0?).
2. Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a continuous function. Define the function $F : [c, d] \to \mathbb{R}$ by
$$F(t) = \int_a^b f(x, t)\, dx.$$
Show that F is a continuous function.
3. With the notation as in the previous problem suppose that $f_t = \frac{\partial f}{\partial t}$ is defined and continuous for all points in D (that is for each point (x_0, t_0) think of $f(x_0, t)$ as a function of just t and take its derivative with respect to t). Then the function F from the last problem is differentiable on $[c, d]$ and
$$F'(t) = \int_a^b f_t(x, t)\, dx.$$
4. Use the ideas in the last problem to integrate
$$\int_0^1 \frac{x^t - 1}{\ln x}\, dx.$$
You can use that $\frac{d}{dt} a^t = (\ln a) a^t$. Hint: First notice that this is not an improper integral (the integrand is continuous on $[0, 1]$. Think of the integral as a function $f(t)$. Compute the derivative and then try to recover $f(t)$.
5. Let $f_n = \frac{x^n}{1+x^n}$ on the interval $[0, 2]$. Determine what function the f_n converge to. Is the convergence uniform?
6. Find a sequence of functions that are everywhere discontinuous on $[0, 1]$ but converge uniformly to a continuous function.
7. Suppose that $\{f_n\}$ is a sequence of bounded functions on a set $A \subset \mathbb{R}$ that converge uniformly on A to f. Show that if each of the f_n are bounded then f is bounded.
8. Consider $f_n = \frac{nx}{1+nx^2}$ for $x \in [0, \infty)$. Show that the f_n are bound. Let f be the point wise limit of the $\{f_n\}$. Show that f is not bounded. (Form this and the last problem we see that the f_n do not converge uniformly to f.)

   Let $\{f_n\}$ be a sequence of functions on a set $S \subset \mathbb{R}$. We say the sequence is **equicontinuous** on S if for every $\epsilon > 0$ there is some $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $|x - y| < \delta$ with x and y in S and for all n.
9. Show that if $\{f_n\}$ is an equicontinuous sequence of functions converges point wise to f, then f is uniformly continuous.
10. Show that if $\{f_n\}$ is a sequence of continuous functions that converge uniformly on a compact set, then the sequence is equicontinuous on that set.